Some Notes on the Smooth Base Change Theorem

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Contents

Disclaimer: There are many subtleties that are missing, especially those related to finiteness conditions on complexes of sheaves.

1 Introduction

Theorem 1.1 (Smooth base change)**.** *Let $f : X \to S$ be a smooth morphism, $g : S' \to S$ be a (quasi-compact) morphism of schemes, and $\mathscr{F}$ a sheaf of $\mathbb{Z}/n$-*

modules on S', where n is invertible on S,

$$\begin{array}{ccc} X' = X \times_S S' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

then we have an isomorphism

$$f^* R^q g_* \mathscr{F} \to R^q g'_* f'^* \mathscr{F}.$$

Remark 1.2. By taking limit, the same statement still holds if we take $\mathscr{F}$ to be a $\mathbb{Q}_l$-sheaf on S'. Thus, we only need to care about the "torsion" case.

Remark 1.3. The isomorphism above comes from isomorphisms on the level of complexes. Indeed, if $L \in D^b(S')$, then the derived-categorical version of this version is

$$f^* R g_* L \to R g'_* f'^* L$$

is a quasi-isomorphism (i.e. isomorphism in the derived category). This is the form that we will take in this talk. For brevity, we have written $D^b(S) = D^b(S, \mathbb{Z}/n)$.

Remark 1.4. In fact, the following stronger statement is true. Namely, if we take $K \in D^b(X)$ such that all cohomology sheaves are locally constant, then we have the following isomorphism (in the derived category)

$$K \otimes^L f^* R g_* L \cong R g'_* (g'^* K \otimes^L f'^* L).$$

Remark 1.5. When the map g is proper then regardless of whether f is smooth or not, we have the following

$$K \otimes^L f^* R g_* L \xrightarrow{\cong} K \otimes^L R g'_* f'^* L \xrightarrow{\cong} R g'_* (g'^* K \otimes^L f'^* L),$$

where the first isomorphism is the proper base change theorem and the second isomorphism is projection/pull-push formula for proper morphisms.

Note that the map

$$K \otimes^L f^* R g_* L \to R g'_* (g'^* K \otimes^L f'^* L)$$

always exists. It's the isomorphism that we are seeking.

2 Local Acyclicity

We will consistently use the following notation.

Notations: $f : X \to S$ morphism of schemes, x a point in X, $s = f(s)$ a point in S. $\widetilde{X}_x = \operatorname{Spec} \mathcal{O}^{\mathrm{sh}}_{X,x}$, $\widetilde{S}_s = \operatorname{Spec} \mathcal{O}^{\mathrm{sh}}_{S,s}$. Usually, we denote $\bar{t}$ an arbitrary geometric point of $\widetilde{S}_s$. $K \in D^b(X)$. We will also frequenly use the following diagram:

$$\begin{array}{ccccc} X_{\bar{s}} & \xrightarrow{\tilde{i}} & X \times_S \widetilde{S}_s & \xleftarrow{\tilde{j}} & X_{\bar{t}} \\ \downarrow{\scriptstyle f_s} & & \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f_t} \\ \bar{s} & \xrightarrow{i} & \tilde{S}_s & \xleftarrow{j} & \bar{t} \end{array}$$

Finally, unless otherwise specified, all geometric points are algebraic. For example, if $\operatorname{Spec} K \to S$ is a geometric point of S whose image is t, then K is algebraic over $k(t)$. In other words, $K \subset \overline{k(t)}$.

2.1 Motivation and Preliminary Properties

Étale locally, a smooth morphism is like a map $\mathbb{A}^n_S \to S$ and hence, we expect that cohomology behaves nicely from fibers to fibers. We wish to use this property to establish the smooth base change theorem. We will now formulate this notion precisely.

Definition 2.1. *f is locally acyclic at x with respect to K if the following natural map is an isomorphism (take appropriate pull-back of K to various spaces to make sense of this) for any geometric point $\bar{t}$ of $\widetilde{S}_{\bar{s}}$.*

$$R\Gamma(\widetilde{X}_x, K) \cong R\Gamma(\tilde{X}_x \times_{\tilde{S}_s} \bar{t}, K).$$

Equivalently, the following adjunction map is an isomorphism

$$(\tilde{i}^* K)_{\bar{x}} \to (\tilde{i} R\tilde{j}_* \tilde{j}^* K)_{\bar{x}}.$$

Definition 2.2. *f is locally acyclic with respect to K if it's locally acyclic with respect to K at all x.*

f is universally locally acyclic (ULA) with respect to K if the pull-back of f and K via any morphism $S' \to S$ are still acyclic (with respect to each other).

Remark 2.3. We see at once that f is locally acyclic with respect to K if and only if for all points $s \in S$ and for all geometric point $\bar{t}$ of $\widetilde{S}_{\bar{s}}$, the following natural morphism is an isomorphism

$$\tilde{i}^* K \to \tilde{i}^* R\tilde{j}_* \tilde{j}^* K.$$

We will now see how we can insert the complex L as in the base change theorem.

Proposition 2.4. *Suppose f is locally acyclic with respect to K then for any $L \in D^b(\bar{t})$ (basically, just a complex of $\mathbb{Z}/n$-modules), we have an isomorphism*

$$\tilde{i}^*(K \otimes^L \tilde{f}^* R j_* L) \to \tilde{i}^* R\tilde{j}_*(\tilde{j}^* K \otimes^L f_t^* L).$$

Proof. First, note that the natural map $i^*L \to i^*Rj_*j^*L$ is an isomorphism when L is a complex with locally constant (hence constant) cohomology sheaves. This can be shown by induction on the length of the complex (using truncation). When the complex is of length 1, this can be seen easily from the fact that j_* is exact (using the fact that $\bar{t}$ is a geometric point, and hence, has no higher cohomology).

Thus, if we view L to be a complex over $\tilde{S}_{\bar{s}}$ (just a constant complex), then

$$\begin{aligned}
\tilde{i}^*(K \otimes^L \tilde{f}^* R j_* L) &\cong \tilde{i}^* K \otimes^L \tilde{i}^* \tilde{f}^* R j_* L \\
&\cong \tilde{i}^* K \otimes^L f_s^* i^* R j_* L \\
&\cong \tilde{i}^* K \otimes^L f_s^* i^* R j_* j^* L \\
&\cong \tilde{i}^* K \otimes^L f_s^* i^* L \\
&\cong \tilde{i}^* K \otimes^L \tilde{i}^* \tilde{f}^* L.
\end{aligned}$$

But since $\tilde{i}^* K \cong \tilde{i}^* R\tilde{j}_* \tilde{j}^* K$ by acyclicity assumption, we get

$$\begin{aligned}
\tilde{i}^* K \otimes^K \tilde{i}^* \tilde{f}^* L &\cong \tilde{i}^*(R\tilde{j}_* \tilde{j}^* K \otimes^L \tilde{f}^* L) \\
&\cong \tilde{i}^* R\tilde{j}_*(\tilde{j}^* K \otimes^L \tilde{j}^* f^* L)
\end{aligned}$$

(pull-push formula for complexes L with $H^i(L)$ are locally constant)

$$\cong \tilde{i}^* R\tilde{j}_*(\tilde{j}^* K \otimes^L f_t^* L).$$

□

Remark 2.5. Note to self: the projection formula for proper map is done by looking at stalks. At the level of stalk, proper base change allows us to only look at the geometric fibers. But now, the base of each geometry fiber is just a point. Thus, we are done if we can show projection formula where the complex in the base is such that all cohomology sheaves are locally constant. This statement is proved directly, and it's just formal.

Remark 2.6. Note to self: There is some subtlety about needing $R\tilde{j}_*$ to be a bounded chain complex. This is, however, satisfied by a theorem. (**To return.**)

We have the following easy consequence of the proposition above.

Corollary 2.7. *Suppose f is locally acyclic wrt. K. Let $\bar{t}$ be a geometric point of S,*

$$\begin{CD} X_{\bar{t}} @>{\gamma'}>> X \\ @V{f'}VV @VV{f}V \\ \bar{t} @>{\gamma}>> S \end{CD}$$

then for all $L \in D^b(\bar{t})$ (a complex of $\mathbb{Z}/n$-modules), then the following natural map is an isomorphism

$$K \otimes^L f^* R\gamma_* L \to R\gamma'_*(\gamma'^* K \otimes^L f'^* L).$$

Proof. Basically, just use the fact that two sheaves are isom if there is a map between them that induces isomorphisms on all stalks. □

2.2 Link to the Base Change Map

We will now show that the base-change map is an isomorphism when the morphism is universally locally acyclic (ULA). First, we prove for the case where g is an open immersion.

Proposition 2.8. *If f is locally acyclic wrt. K then for any open immersion $S' \to S$ and for any $L \in D^b(S')$,*

$$\begin{CD} X' @>{g'}>> X \\ @V{f'}VV @VV{f}V \\ S' @>{g}>> S \end{CD}$$

the following natural map is an isomorphism

$$K \otimes^L f^* Rg_* L \to Rg'_*(f'^* K \otimes^L g'^* L).$$

In other words, the base change map is an isomorphism for f and K.

Proof. This is a standard devissage argument. First, let $\bar{t}$ be a geometric point of S', and hence, of S as well, since S' is an open subscheme of S.

$$\begin{CD} X'_{\bar{t}} @>{j'}>> X' @>{g'}>> X \\ @V{f_{\bar{t}}}VV @V{f'}VV @VV{f}V \\ \bar{t} @>{j}>> S' @>{g}>> S \end{CD}$$

We will first show the isomorphism for the case where $L = Rj_*L'$. Then, by applying corollary 2.7 for the square on the left and the big rectangle, we have

$$\begin{aligned} K \otimes^L f^* Rg_* L &\cong K \otimes^L f^* Rg_* Rj_* L' \\ &\cong K \otimes^L f^* R(g \circ j)_* L' \\ &\cong R(g' \circ j')_*((g' \circ j')^* K \otimes^L f_{\bar{t}}^* L') \\ &\cong Rg'_* Rj'_* (j'^* g'^* K \otimes^L f_{\bar{t}}^* L') \\ &\cong Rg'_* (g'^* K \otimes^L f'^* Rj_* L') \\ &\cong Rg'_* (g'^* K \otimes^L f'^* L). \end{aligned}$$

In particular, this is true when $\bar{t}$ is a geometric point over a closed point of S'. This serves as the first step of our induction. Suppose we have an isomoprhism for any complex L supported on a proper closed subscheme of S', we will now prove that it's also true for any complex L in general. Without loss of generality, we assume that S' is irreducible. Let $\bar{\eta}$ be a geometric point of the generic point of S' and $\gamma : \bar{\eta} \to S'$, then for any complex $L \in D^b(S')$, we have the following exact triangle

$$L \longrightarrow R\gamma_* \gamma^* L \longrightarrow P \longrightarrow \cdots$$

Since $\gamma^* R\gamma_* \gamma^* L \cong \gamma^* L$, $\gamma^* P \cong 0$ and hence, P is suported on a proper closed subscheme of S'. Thus, we get the isomorphism for P. The isomorphism for $R\gamma_* \gamma^* L$ is from corollary 2.7 above. Thus, we are done. □

Remark 2.9. There is another way to do devissage for this case. First, we can assume that L concentrates at one degree, i.e. it's a sheaf. Moreover, we can assume that this sheaf is constructible. Thus, we can embed our sheaf into $\oplus j_* \mathbb{Z}/n$ and do a 5-lemma argument as in the proof of the proper base change theorem.

We will now show the isomorphism in general.

Theorem 2.10. *If f is universally locally acyclic wrt. K then for any morpshim $g : S' \to S$ (that is quasi-compact and quasi-separated) and for any $L \in D^b(S')$, the following natural is an isomorphism*

$$K \otimes^L f^* Rg_* L \to Rg'_* (g'^* K \otimes^L f'^*).$$

Proof. Using an affine open cover of S' and an argument with the Cech to derived cohomology spectral sequence, we can reduce to the case where S' is affine. By passage to limits, we can assume that this affine subscheme is of finite type over S. But now, we can compactify S':

$$\begin{array}{ccccc} X' & \xrightarrow{j'} & \overline{X'} & \xrightarrow{\bar{g}'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle \bar{f}} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{j} & \overline{S'} & \xrightarrow{\bar{g}} & S \end{array}$$

where j is an open immersion, and $\bar{g} \circ j = g$.

Now, we have

$$\begin{aligned} K \otimes^L f^* R g_* L &\cong K \otimes^L f^* R\bar{g}_* R j_* L \\ &\cong R\bar{g}'_*(\bar{g}'^* K \otimes^L \bar{f}^* R j_* L) && \text{(proper base change)} \\ &\cong R\bar{g}'_* R j'_*(j'^* g'^* K \otimes^L f'^* L) && \text{(by ULA and theorem)} \\ &\cong R g'_*(g^* K \otimes^L f'^* L). \end{aligned}$$

□

3 Local Acyclicity of Smooth Morphisms

From what we have shown in the previous section, to prove smooth base-change theorem, it suffices to show the following.

Theorem 3.1. *Let $f : X \to S$ be a smooth morphism, and let n be an integer invertible on S, then f is universally locally acyclic wrt. $\mathbb{Z}/n$.*

Before proving the theorem, we note the following direct consequence.

Corollary 3.2. *Let $f : X \to S$ be a smooth morphism and let n be an integer invertible on S. Then for any complex $K \in D^b(X)$ whose cohomology sheaves are locally constant on X, f is universally locally acyclic wrt. K.*

Proof (Sketch). It suffices to prove for a single locally constant sheaf (and then use induction on length). But for a locally constant sheaf, étale-locally, the sheaf is just a constant sheaf, which can be thought of as the pull-back of a constant sheaf M on the base. Using the smooth base change theorem above, we basically reduce to showing that

$$i^* M \cong i^* R j_* M.$$

But this has been shown before, and hence, we are done. □

We will now sketch the proof of theorem 3.1. First, note that it suffices to prove local acyclicity (instead of universal local acyclicity) since smoothness is preserved under base change.

3.1 Reduction Step

We will now try to reduce to the following case: $S = \mathrm{Spec}A$, where A is a strict Henselian ring, $X = \mathrm{Spec}A\{T\}$, where $A\{T\}$ denotes the strict Henselization of $A[T]$ at $(\mathfrak{m}, T)$ with $\mathfrak{m}$ being the maximal ideal of A.

First, we observe that the composition of two morphisms that are acyclic wrt. to $\mathbb{Z}/n$ is also acyclic wrt. $\mathbb{Z}/n$. Now, the situation is local on both S on X. Thus, we can assume that $S = \mathrm{Spec}A$, where A is a strict Henselian ring. Now, since X is smooth over S, after shrinking X, we can factor $X \to S$ as $X \to \mathbb{A}^n_S \to S$ where $X \to \mathbb{A}^n_S$ is an etale morphism. Since acyclicity is clear for étalemorphisms (as the fiber over any formal disc around a geometric point is just a disjoint union of the same thing, it suffices to prove for the case $X = \mathbb{A}^n_S$. But now, since we can factor $\mathbb{A}^n_S \to \mathbb{A}^{n-1}_S \to \cdots \to \mathbb{A}^1_S \to S$, it suffices to prove for the case $X = \mathbb{A}^1_S$. Since the situation is local over X, it suffices to do it for the case where $X = \mathrm{Spec}A\{T\}$. Moreover, we can also assume that the geometric point $\bar{t}t$ of S lies over the generic point t of S and S is normal.[1]

3.2 End of the Proof

By what we have said above, we only need to consider the following case: $X = \mathrm{Spec}A\{T\}$ and $S = \mathrm{Spec}A$ with A normal, strict Henselian, $\bar{t}$ a geometric point lying over the generic point of S.

We need to prove the following:

Theorem 3.3.

$$H^i(X \times_S \bar{t}) \cong \begin{cases} \mathbb{Z}/n, & \text{when } i = 0, \\ 0, & \text{when } i > 0. \end{cases}$$

First, since we are working with (limits of open curves), we only care about $i = 0, 1$ since for $i > 1$, cohomology groups vanish. This is still quite difficult and it relies on tricky results in commutative algebra.

$i = 0$. Essentially, all we need to do is to show that $X \times_S t'$ is connected for any separable extension $k(t')$ of $k(t)$ inside $k(t)$ (since cohomology commutes with projective limits of schemes). In other words, we want to show that $\mathrm{Spec}A\{T\} \otimes_A k(t')$ is connected. Note that $A\{T\} \subset A[[T]]$, and since $k(t')$ is flat over A, we have $A\{T\} \otimes_A k(t') \subset A[[T]] \otimes_A k(t') \subset k(t')[[T]]$ where the last

[1] This is basically because pulling back to $\bar{t}$ is the same as pulling back to the normalizer of $\overline{\{t\}}$ in $k(t)$ first and then pulling back to $k(\bar{t})$. There's something about excellent ring that says that the normalizer is finite over S and is also strictly Henselian.

inclusion is due to the fact that $A \subset k(t) \subset k(t')$. Since $k(t')[[T]]$ is integral, so is $A\{T\} \otimes_A k(t')$.
(In SGA $4\frac{1}{2}$, there's some argument that goes like this, which I don't fully understand. Let A' be the normalizer of A inside $k(t')$, then some result of Henselian ring says that $A\{T\} \otimes_A A' \cong A'\{T\}$, which is a normal strict Henselian ring, which is, in particular, an integral domain. Thus, we can conclude the same thing for $A\{T\} \otimes_A k(t') \cong A\{T\} \otimes_A A' \otimes_{A'} k(t')$ since $k(t')$ is just the field of fractions of A'. But integrality implies connectedness and we are done.)

$i = 1$. For H^1, we want to show the following

Proposition 3.4. *All Galois étale covers of $X \times_S \bar{t}$ of degree n are trivial (where n is invertible in S).*

In other words, if $\widetilde{X}_{\bar{t}}$ be an étale Galois cover of $X_{\bar{t}}$, then it's trivial. This is a bit more involved. We need the following result by Abhyankar. The argument uses some depth argument. I haven't studied it carefully yet, but the upshot is that it suffices to prove things for an open set whose complement has codim ≥ 2 (EGA IV, vol 4).

Theorem 3.5. *Let $S = \operatorname{Spec} A$ be a strictly Hensenlian DVR with uniformizer π and generic point t, and X an irreducible scheme which is smooth with relative dimension 1 over S. Let $\widetilde{X}_t$ be an Galois étale cover of X_η of degree n invertible in S, and $S_1 = \operatorname{Spec}(A[\pi^{1/n}])$. Denote the subscript $_1$ be the base change from S to S_1. Then, $\widetilde{X}_{1\eta}$ can be extended to an étale cover of X_1.*

Using this result, we can assume that (the price is to enlarge $k(t)$, which is ok, since we actually base change to $k(\bar{t})$ in the end) $\widetilde{X}_{\bar{t}}$ actually comes from V', an étale Galois cover of $V = X_U$, where U is an open subscheme of S whose complement has codimension ≥ 2. Further enlarging $k(t)$, we can also assume that over $T = 0$, this cover is trivial.

Now, let $B = \Gamma(V', \mathcal{O}_{V'})$, then since V' is the preimage of V in $\operatorname{Spec} B$,[2] it suffices to prove that B is finite étale over $A\{T\}$ (and hence, decompose into direct product of copies of $A\{T\}$ since $A\{T\}$ is strictly Henselian). Let $\hat{X} = \operatorname{Spec} A[[T]]$ and denote $\hat{}$ the base change of everything from X to $\hat{X}$. $\hat{X}$ is

[2]This probably goes as follows. By Zariski's main theorem, we know that there is an open immersion $V' \to V''$, where V'' is finite over $\operatorname{Spec} A\{T\}$. Moreover, since $V' \to V$ is finite, we can choose V'' such that the preimage of V inside V'' is just V'. The only problem left is to show that $\Gamma(V'', \mathcal{O}_{V''}) = \Gamma(V', \mathcal{O}_{V'}) = B$. Since the complement of V in $\operatorname{Spec} A\{T\}$ has codim ≥ 2, the same statement applies to V' in V''. If this implies some depth ≥ 2 then we should have our desired conclusion.

faithfully flat over X. One then have $\Gamma(\hat{V}', \mathcal{O}_{\hat{V}'}) \cong B \otimes_{A\{T\}} A[[T]] =: \hat{B}$, and it suffices to see that this ring $\hat{B}$ is finite étale over $A[[T]]$ since étaleness is a local property in the flat topology.

Let V_m (resp. V'_m) be the subscheme of $\hat{V}$ (resp. $\hat{V}'$) defined by $T^{m+1} = 0$. By assumption, V'_0 is a trivial cover of V_0, and hence, so are the covers V'_m/V_m, since the étale topology is insensitive to nilpotents.[3] We have the following map

$$\varphi : \Gamma(\hat{V}', \mathcal{O}_{\hat{V}'}) \to \varprojlim_m \Gamma(V'_m, \mathcal{O}_{V'_m}) = (\varprojlim_m \Gamma(V_m, \mathcal{O}_{V_m}))^n.$$

We have $\Gamma(V_m, \mathcal{O}_{V_m}) \cong A[T]/(T^{m+1})$ (some depth ≥ 2 thing), and φ is a homomorphism of $\hat{B} \to A[[T]]^n$. Over U, this gives n distinct sections of $\hat{V}'/\hat{V}$: $\hat{V}'$ is hence just n copies of $\hat{V}$. The complement of $\hat{V}$ in $\hat{X}$ is of codimension 2 (and things are normal), hence, we indeed have $\hat{B} = A[[T]]^n$, and we are done. Upshot: only need to prove things for an open subscheme whose complement has codim ≥ 2.

4 Applications

In this section, unless otherwise specified, $f : X \to S$ be a morphism of schemes, K a complex of $\mathbb{Z}/n$-sheaves on X, $s, t, \bar{s}, \bar{t}, i, j, \bar{i}, \bar{j}$ are as above.

Using the natural map $\bar{i}^*K \to \bar{i}^*R\bar{j}_*\bar{j}^*K$, we have the following sequences of maps:

$$R\Gamma(X_{\bar{s}}, \bar{i}^*K) \to R\Gamma(X_{\bar{s}}, \bar{i}^*R\bar{j}_*\bar{j}^*K) \leftarrow R\Gamma(X \times_S \tilde{S}_{\bar{s}}, R\bar{j}_*\bar{j}^*K) \leftarrow R\Gamma(X_{\bar{t}}, \bar{j}^*K).$$

The last arrow is always an isomorphism (basically because push forward preserves injectives). When f is locally acyclic wrt. K, then the first arrow is also an isomorphism. Hence, we get the cospecialization map:

$$R\Gamma(X_{\bar{t}}, \bar{j}^*K) \to R\Gamma(X_{\bar{s}}, \bar{i}^*K).$$

Proposition 4.1. *Let $f : X \to S$ be a locally acyclic morphism wrt. $\mathbb{Z}/n$. Suppose the cospecialization map*

$$R\Gamma(X_{\bar{t}}, \mathbb{Z}/n) \to R\Gamma(X_{\bar{s}}, \mathbb{Z}/n)$$

is an isomorphism for any $\bar{s}, \bar{t}$, then for any $L \in D^b(S, \mathbb{Z}/n)$, we have

$$(Rf_*f^*L)_{\bar{s}} \cong R\Gamma(X_{\bar{s}}, f^*L).$$

[3]More precisely, we can just lift all the sections that trivialize this étale covering using lifting property of étale morphisms (formal smooth etc.).

Proof (Sketch). We only need to prove for sheaves. The result is immediate for the case where $L = j_*\mathbb{Z}/n$, using smooth base change. For a constructubible thing, we can embed to $\oplus j_*\mathbb{Z}/n$ and argue as usual. Or alternative, did the devisage as above. Thus, we are done. □

Corollary 4.2. *Let $f : \mathbb{A}^n_S \to S$ be the canonical projection then for any $L \in D^b(S)$, the canonical morphism is an isomorphism in the derived category*

$$L \to Rf_*f^*L.$$

Proof (Sketch). It suffices prove for the case $n = 1$ and L is a sheaf. Moreover, we only need to check the stalks, i.e. we want to show that

$$L_{\bar{s}} \cong (Rf_*f^*L)_{\bar{s}}.$$

But we note that the second morphism and the composition of the two morphisms in the following sequence of maps are isomorphisms (by the corollary above and an explicit computation of the cohomology of $\mathbb{A}^n$):

$$L_{\bar{s}} \to (Rf_*f^*L)_{\bar{s}} \to R\Gamma(\mathbb{A}^n_{\bar{s}}, f^*L).$$

So, the remaining one $L_{\bar{s}} \to (Rf_*f^*L)_{\bar{s}}$ has to be an isomorphism as well. □

Corollary 4.3. *Let $f : X \to S$ be a proper morphism and smooth morphism, then for any locally constant sheaf $\mathscr{F}$ on X, $R^if_*\mathscr{F}$ is a locally constant sheaf on S.*

Proof (Sketch). By a theorem of Deligne, we know that R^if_* are all constructible. Since all cospecialization maps are isomorphisms (by using both smooth base change and proper base change), all these sheaves must be locally constant. □

5 Nearby and Vanishing Cycles

5.1 Motivation and Definition

In this section, S is the spectrum of a strict Henselian DVR (a trait, in the language of SGA) and X be a scheme over S. We keep the same notation as above, but replace t and $\bar{t}$ by η and $\bar{t}$ respectively, where η is the generic point of S.

Let $K \in D^b(X)$, then we have seen above that the natural morphism plays an important role in the whole theory of acyclicity

$$\tilde{i}^*K \to \tilde{i}^*R\tilde{j}_*\tilde{j}^*K.$$

The nearby cycle associated to K and f is defined by

$$R\Psi_f K = \tilde{i}^* R\tilde{j}_* \tilde{j}^* K.$$

When f is locally acyclic wrt. K then this morphism is an isomorphism. In general, it is not and we can complete it to an exact triangle. The cone is defined as the vanishing cycle complex and we have the following exact triangle

$$\tilde{i}^* K \to R\Psi_f K \to R\Phi_f K \to \tilde{i}^* K[1] \to \cdots$$

From what we have seen, the following two propositions are obvious:

Proposition 5.1. *When f is smooth and when $K \in D^b(X)$ such that all cohomology sheaves are locally constant then $R\Phi_f K$ vanishes.*

Proof. Clear. □

Proposition 5.2. *When f is proper and $K \in D^b(X)$ then the following natural morphism is an isomorphism*

$$R\Gamma(X_{\bar{\eta}}, \tilde{j}'^* K) \to R\Gamma(X_{\bar{s}}, R\Psi_f K).$$

Remark 5.3. The catch phrase: cohomology of the generic fiber is isomorphic to the cohomology of the nearby cycles on the special fiber.

Proof. Direct from the proper base change theorem. □

5.2 A Sketch of Monodromy of Lefschetz Pencil

To be written.

6 Reference

SGA $4\frac{1}{2}$.