TOPIC PROPOSAL
HILBERT AND QUOT SCHEMES WITH SOME BASIC APPLICATIONS

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ABSTRACT. Hilbert and Quot schemes are the basic blocks to construct other moduli spaces. In this note, after presenting preliminary results, we will sketch their constructions. As an application, we will finish with the construction of the Hom schemes. All the ideas and results presented here are written in [Gro], [FGI+06], [Mum66] and [HL10].

1. INTRODUCTION

1.1. **Conventions.** Our base scheme S is always noetherian and Sch_S denotes the category of locally noetherian schemes over S.

1.2. **Definitions.** The Hilbert and Quot schemes are defined to be the representing objects for the corresponding functors defined below.

Definition 1.1 (Family of quotients)**.** *Let $X \to S$ be a finite type scheme over S and E a coherent sheaf on X. For any $T \to S$ in Sch_S, a family of quotients of E parametrized by T means a pair $(\mathscr{F}, q)$, where*

(i) $\mathscr{F}$ is a coherent sheaf on $X_T = X \times_S T$ such that the schematic support of $\mathscr{F}$ is proper over T and $\mathscr{F}$ is flat over T.
(ii) $q : E_T \to \mathscr{F}$ is a surjective homomorphism of $\mathscr{O}_{X_T}$-modules, where E_T is the pull back of E via $X_T \to X$.

Two such families $(\mathscr{F}, q)$ and $(\mathscr{F}', q')$ parametrized by T are said to be equivalent if there is an isomorphism of $\mathscr{O}_{X_T}$-modules $\varphi : \mathscr{F} \to \mathscr{F}'$ that makes the obvious diagram commute; namely $q' = \varphi \circ q$. In other words, they are equivalent if and only if $\ker q = \ker q'$. The equivalence class of $(\mathscr{F}, q)$ will be denoted by $\langle \mathscr{F}, q \rangle$.

Definition 1.2 (The Quot functor)**.** *Let E, X, S be as above. The functor $\mathfrak{Quot}_{E/X/S} : \mathrm{Sch}_S \to \mathrm{Sets}$ is defined as follows*

$$\mathfrak{Quot}_{E/X/S}(T) = \{\text{All } \langle \mathscr{F}, q \rangle \text{ parametrized by } T\}$$

for any $T \to S$ in Sch_S.

Definition 1.3 (The Hilbert functor)**.** *Let X, S be as above. The functor $\mathfrak{Hilb}_{X/S} : \mathrm{Sch}_S \to \mathrm{Sets}$ is defined as follows*

$$\mathfrak{Hilb}_{X/S} = \mathfrak{Quot}_{\mathscr{O}_X/X/S}.$$

Equivalently, $\mathfrak{Hilb}_{X/S}(T) = \{Y \subset X_T \mid Y \text{ is flat and proper over } T\}$.

Definition 1.4 (Hilbert and Quot schemes)**.** *Let E, X, S be as above. The Hilbert scheme $\mathrm{Hilb}_{X/S}$ is the scheme representing the functor $\mathfrak{Hilb}_{X/S}$, and the Quot scheme $\mathrm{Quot}_{E/X/S}$ is the scheme representing the functor $\mathfrak{Quot}_{E/X/S}$ (if they exist).*

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1.3. **Stratification by Hilbert Polynomials.** Let X be a projective scheme over a field k and $\mathscr{F}$ a coherent sheaf on X. Let $\mathscr{O}(1)$ be a very ample line bundle on X, then we know that $\Phi(m) = \chi(\mathscr{F}(m))$ is a polynomial, called the Hilbert polynomial.

When X is projective over S, $\mathscr{F}$ is a coherent sheaf on X, flat over S, and $\mathscr{L} = \mathscr{O}(1)$ a very ample line bundle on X/S, then the polynomial $\Phi_s(m) = \chi(\mathscr{F}(m)|_{X_s})$ is locally constant on S. Thus, if we let $\mathfrak{Quot}^{\Phi,\mathscr{L}}_{E/X/S}$ to be the functor associating to any T the set of all $\langle \mathscr{F}, q\rangle$ such that for each $t \in T$, the Hilbert polynomial of the restriction $\mathscr{F}_t$ is Φ, then we have have a natural decomposition

$$\mathfrak{Quot}_{E/X/S} = \coprod_{\Phi \in \mathbb{Q}[\lambda]} \mathfrak{Quot}^{\Phi,\mathscr{L}}_{E/X/S},$$

where the natural inclusion of functors $\mathfrak{Quot}^{\Phi,\mathscr{L}}_{E/X/S} \to \mathfrak{Quot}_{E/X/S}$ is representable, and in fact a closed and open embedding, for each Φ. Thus, it suffices to represent $\mathfrak{Quot}^{\Phi,\mathscr{L}}_{E/X/S}$.

1.4. **The General Strategy.** A basic general strategy to represent a functor F is to find a scheme G and a representable morphism $F \to G$ of functors (where by G, we actually mean the functor h_G represented by G). In the case of Quot schemes, our G will be certain Grassmannian. The morphism $F \to G$ will essentially be given by Castelnuovo-Mumford regularity, and its representability will be given by flattening stratification.

1.5. **The Grassmannians.** Let S be a Noetherian scheme as usual, and E a locally free sheaf of rank n over S, then $\mathfrak{Grass}(E,r) = \mathfrak{Quot}^{r,\mathscr{O}_S}_{E/S/S}$. In other words, $\mathfrak{Grass}(E,r)$ classifies locally free quotients of E of rank r. Since the problem is local over S, to represent $\mathfrak{Grass}(E,r)$, it suffices to do for the case where E is free. When E is free, the construction is similar to that of $\mathbb{P}^d$, where the action of $\mathbb{G}_m$ on $\mathbb{A}^{d+1}\backslash\{0\}$ is replaced by the action of GL_r on the space of $n \times r$ matrices of rank r. In the end, the arrive at the scheme $\mathrm{Grass}(n,r)$ representing $\mathfrak{Grass}(n,r)$, which is projective and smooth over S with relative dimension $r(n-r)$. An explicit embedding $\mathrm{Grass}(n,r) \to \mathbb{P}(\bigwedge^r E)$ is given by

$$\begin{aligned} \mathrm{Grass}(n,r)(T) &\longrightarrow \mathbb{P}(\textstyle\bigwedge^r E)(T) \\ (E_T \twoheadrightarrow \mathscr{F}) &\longmapsto (\textstyle\bigwedge^r E_T \twoheadrightarrow \det \mathscr{F}). \end{aligned}$$

2. Castelnuovo-Mumford Regularity

Roughly speaking, Castelnuovo-Mumford regularity gives us some bound on the dimension of the Grassmannian that we want to embed our Quot scheme into.

Definition 2.1. *Let k be a field, $\mathscr{F}$ a coherent sheaf on $\mathbb{P}^n_k$, and m an integer. The sheaf $\mathscr{F}$ is said to be m-regular if*

$$H^i(\mathbb{P}^n, \mathscr{F}(m-i)) = 0, \quad \forall i \geq 1.$$

Lemma 2.2. *If $\mathscr{F}$ is an m-regular sheaf on $\mathbb{P}^n$ then*

(i) $\mathscr{F}$ *is m'-regular for all $m' \geq m$.*
(ii) $H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1)) \otimes H^0(\mathbb{P}^n, \mathscr{F}(r)) \to H^0(\mathbb{P}^n, \mathscr{F}(r+1))$ *is surjective whenever $r \geq m$.*
(iii) $\mathscr{F}(r)$ *is generated by its global section when $r \geq m$.*

Proof (Sketch). The proof is an induction on the dimension of $\mathbb{P}^n$, i.e. on the number n. Since cohomology base changes correctly under field extension (since all fields extensions are flat), we can assume that k is infinite, in which case, we can assume that there exists a hyper-plane H

defined by $\alpha = 0$ that misses all the finitely many associated points of $\mathscr{F}$. The induction is then done via the long exact sequence associated to the following exact sequence

$$0 \longrightarrow \mathscr{F}(m-i-1) \xrightarrow{\cdot\alpha} \mathscr{F}(m-1) \longrightarrow \mathscr{F}_H(m-i) \longrightarrow 0.$$

□

The following result gives a bound needed for the construction of the Quot scheme.

Theorem 2.3. *For any non-negative integers p and n, there exists a polynomial $F_{p,n}$ in $n+1$ variables with integral coefficients, which has the following property:*

Let k be any field, and let $\mathbb{P}^n$ denote the n-dimensional projective space over k. Let $\mathscr{F}$ be any coherent sheaf on $\mathbb{P}^n$, which is isomorphic to a subsheaf of $\bigoplus^p \mathscr{O}_{\mathbb{P}^n}$. Suppose the Hilbert polynomial of $\mathscr{F}$ is written as

$$\chi(\mathscr{F}(r)) = \sum_{i=1}^{n} a_i \binom{r}{i}$$

where $a_i \in \mathbb{Z}$.

Then $\mathscr{F}$ is m-regular, where $m = F_{p,n}(a_0, \ldots, a_n)$.

3. The Grothendieck Complex

Besides applications to the base change and semi-continuity theorems, the Grothendieck complex, which is a presentation of $Rf_*\mathscr{F}$ as a bounded perfect complex, can be used to construct the moduli spaces of morphisms of sheaves.

Theorem 3.1. *Let $\pi : X \to S$ be a proper morphism of noetherian schemes where $S = \operatorname{Spec} A$, and let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module which is flat over S. Then the following complex is isomorphic to $Rf_*\mathscr{F}$*

$$0 \longrightarrow K^0 \longrightarrow K^1 \longrightarrow \cdots \longrightarrow K^n \longrightarrow 0,$$

where K_i's are finitely generated projective A-modules. Moreover, for each module M, there is a natural A-linear isomorphism

$$H^p(X, \mathscr{F} \otimes M) \cong H^p(K^\bullet \otimes_A M).$$

Proof (Sketch). Since everything involved are separated, $Rf_*\mathscr{F}$ can be computed using the Čech complex formed by an affine covering of X. This is a complex concentrated on degrees from 0 to n, and all cohomologies have finite ranks, by properness. By a downward inductive process, we can build a complex $K^\bullet$ quasi-isomorphic to it. The fact that $\mathscr{F}$ is flat is needed to make sure that K^0 is projective. □

Theorem 3.2. *Let S be a noetherian scheme and $\pi : X \to S$ a proper morphism. Let $\mathscr{F}$ be a coherent sheaf on X which is flat over S, then there exists a coherent sheaf $\mathscr{Q}$ on S together with a functorial $\mathscr{O}_S$-linear isomorphism*

$$\theta : \pi_*(\mathscr{F} \otimes_{\mathscr{O}_X} \pi^* -) \to \underline{Hom}_{\mathscr{O}_S}(\mathscr{Q}, -)$$

in the category of quasi-coherent sheaves on S. The pair $(\mathscr{Q}, \theta)$ is unique.

Proof (Sketch). Uniqueness comes from abstract nonsense about representable functors. But by uniqueness, we only need to construct $\mathscr{Q}$ locally on S; in particular, we can reduce to the case where $S = \operatorname{Spec} A$. But now, we see easily that $\mathscr{Q} = \widetilde{M}$, where $M = \operatorname{coker}((K^1)^\vee \to (K^0)^\vee)$. □

Definition 3.3. *A linear scheme $\mathbf{V} \to S$ is a scheme of the form $\mathbf{Spec}\operatorname{Sym}_{\mathscr{O}_S} \mathscr{Q}$ where $\mathscr{Q}$ is a coherent sheaf on S.*

We see at once that $\mathbf{V}$ is a group scheme, with zero section given by $0 \in \mathrm{Hom}(\mathscr{Q}, \mathscr{O}_S)$. The image of the zero section is defined by the ideal generated by $\mathscr{Q} \to \mathrm{Sym}_{\mathscr{O}_S}\mathscr{Q}$. We call this scheme $\mathbf{V}_0$.

Theorem 3.4. *Let S be a noetherian scheme and $\pi : X \to S$ a projective morphism. Let $\mathscr{E}$ and $\mathscr{F}$ be a coherent sheaves on X, with $\mathscr{F}$ flat over S. Let $\mathfrak{Hom}(\mathscr{E},\mathscr{F}) : \mathrm{Sch}_S \to \mathrm{Sets}$ be a functor such that $\mathfrak{Hom}(\mathscr{E},\mathscr{F})(T) = \mathrm{Hom}_{\mathscr{O}_{X_T}}(\mathscr{E}_T, \mathscr{F}_T)$. Then, $\mathfrak{Hom}(\mathscr{E},\mathscr{F})$ is representable by a linear scheme.*

Proof (Sketch). When $\mathscr{E}$ is locally free, then for any $f : T \to S$, we have

$$\mathrm{Hom}_{\mathscr{O}_{X_T}}(\mathscr{E}_T, \mathscr{F}_T) = H^0(X_T, (\mathscr{E}^\vee \otimes_{\mathscr{O}_X} \mathscr{F})_T) = \mathrm{Hom}_{\mathscr{O}_S}(\mathscr{Q}, f_*\mathscr{O}_T) = \mathrm{Hom}_{\mathscr{O}_T}(f^*\mathscr{Q}, \mathscr{O}_T) = \mathbf{Spec}\,\mathrm{Sym}\,\mathscr{Q}(T),$$

where $\mathscr{Q}$ is the module associated to $\mathscr{E}^\vee \otimes_{\mathscr{O}_X} \mathscr{F}$ (see theorem 3.2).

Note that we only need to construct our moduli space Zariski locally on S. But when $\mathscr{E}$ is not locally free, we can present $\mathscr{E}$ as a quotient of free sheaves $E_1 \to E_0 \to \mathscr{E} \to 0$ on open subschemes of S. Now, $\mathrm{Hom}(\mathscr{E},\mathscr{F}) = \ker(\mathrm{Hom}(E_0,\mathscr{F}) \to \mathrm{Hom}(E_1,\mathscr{F}))$. Thus, $\mathfrak{Hom}(\mathscr{E},\mathscr{F})$ can be represented as a closed subscheme of $\mathfrak{Hom}(E_0,\mathscr{F})$ defined by the pull back of the zero section of $\mathfrak{Hom}(E_1,\mathscr{F})$. We see easily that this scheme is precisely the linear scheme associated to $\mathrm{coker}(\mathscr{Q}_1 \to \mathscr{Q}_0)$, where $\mathscr{Q}_0$ and $\mathscr{Q}_1$ are the $\mathscr{Q}$-modules associated to E_0 and E_1 respectively (see theorem 3.2). $\square$

Remark 3.5. If $X, S, \mathscr{E}, \mathscr{F}$ are as in the theorem above and $\mathbf{V}$ is the linear scheme representing the functor $\mathfrak{Hom}(\mathscr{E},\mathscr{F})$, then $\mathbf{V}_0 \subset \mathbf{V}$ is where the universal homomorphism vanishes. Thus, if $f \in \mathrm{Hom}_{\mathscr{O}_{X_T}}(\mathscr{E}_T, \mathscr{F}_T) = \mathbf{V}(T)$, then $f^{-1}\mathbf{V}_0$ is a closed subscheme T' of T such that if $U \to T$ is any morphism whose pull-back of f is zero, then $U \to T$ factors through T'.

4. Flattening Stratification

We start with an elementary result linking the flatness of a sheaf and the locally-freeness of its push forward.

Lemma 4.1. *Let $\mathscr{F}$ be a coherent sheaf on $\mathbb{P}^n_S$, then $\mathscr{F}$ is flat over S if and only if there exists an integer N such that $\pi_*\mathscr{F}(r)$ is locally free for all $r \geq n$.*

Proof (Sketch). The lemma is a direct consequence of Serre's vanishing theorem and the base change theorems for coherent cohomology. $\square$

Theorem 4.2. *Let S be a noetherian scheme, $\pi : \mathbb{P}^n_S \to S$ and $\mathscr{F}$ a coherent sheaf on $\mathbb{P}^n_S$. Then, the set I of Hilbert polynomials of $\mathscr{F}|_{\mathbb{P}^n_s}$, where $s \in S$, is a finite set. Moreover, for each $f \in I$, there exists a locally closed subscheme of S_f of S satisfying the following conditions:*

- *(i) The underlying set $|S_f|$ of S_f consists of all points $s \in S$ where the Hilbert polynomial of $\mathscr{F}|_{\mathbb{P}^n_s}$ is f. In particular, $|S_f| \subset |S|$ are disjoint, and $\coprod |S_f| = |S|$.*
- *(ii) Let $S' = \coprod S_f$, and let $i : S' \to S$ be the natural inclusion. Then $i^*\mathscr{F}$ is flat over S'. Moreover, i has a universal property that for any $\varphi : T \to S$, the pull back $\varphi^*\mathscr{F}$ on $\mathbb{P}^n_T$ is flat over T if and only if T factors through S'.*
- *(iii) Let the total ordering on I be such that $f < g$ if and only if $f(n) < g(n), \forall n \gg 0$. Then, $\overline{|S_f|} = \coprod_{g \geq f} |S_g|$.*

Proof (Sketch). The proof starts with the base case where $n = 0$. Here, $\mathbb{P}^n_S = \mathbb{P}^0_S \cong S$ and hence being flat is equivalent to being locally free. Moreover, the Hilbert polynomials are just the rank of the sheaf at a point in this case. Thus, for any point $s \in S$, we can choose a neighborhood U around it such that we have a two-term resolution of $\mathscr{F}$

$$\mathscr{O}_U^m \xrightarrow{d} \mathscr{O}_U^n \longrightarrow \mathscr{F}|_U \longrightarrow 0,$$

where $n = \dim_{k(s)} \mathscr{F} \otimes k(s)$. Using the fact that pull-back is right exact, we see at once that the subscheme of S defined by the ideal generated by entries of the matrix representing d is the desired $S_n \cap U$. These subschemes patch together by uniqueness to give us the locally closed subscheme S_n.

The general case is done by reducing to the base case. Using generic flatness, base change theorems and of course, the fact that S is noetherian, we can show that there is an integer N such that the base change map

$$(\pi_* \mathscr{F}(m))|_s \to H^0(\mathbb{P}^n_s, \mathscr{F}_s(m))$$

is an isomorphism and $H^r(\mathbb{P}^n_s, \mathscr{F}_s(m)) = 0$, for all $m \geq N$. Combined with lemma 4.1, this implies that for any $T \to S$, $\mathscr{F}_T$ is flat if and only if $\pi_{T*}\mathscr{F}_T(m)$ is locally free for all $m \geq N$. This establishes the desired link to the base case, and we are done. □

5. Construction of Quot Schemes

In this section, we will sketch the proof of the following theorem due to Grothendieck.

Theorem 5.1. *Let S be a Noetherian scheme, $\pi : X \to S$ a projective morphism, and $\mathscr{L}$ a relatively very ample line bundle on X. Then, for any coherent $\mathscr{O}_X$-module E, and any polynomial $\Phi \in \mathbb{Q}[\lambda]$, the functor $\mathfrak{Quot}^{\Phi,\mathscr{L}}_{E/X/S}$ is representable by a projective S-scheme $\mathrm{Quot}^{\Phi,\mathscr{L}}_{E/X/S}$.*

We will first sketch the main steps in the proof of the following theorem, which is due to Altman and Kleiman, and then indicate briefly in a remark at the end of this section how we get theorem 5.1 from it.

Theorem 5.2. *Let S be a noetherian scheme, X a closed subscheme of $\mathbb{P}(V) = \mathbf{Proj}\,\mathrm{Sym}_{\mathscr{O}_S} V$ for some vector bundle V on S, $\mathscr{L} = \mathscr{O}_{\mathbb{P}(V)}(1)|_X$, E a coherent quotient sheaf of $\pi^*(W)(v)$, where W is a vector bundle on S and v is an integer. Let $\Phi \in \mathbb{Q}[\lambda]$, then the functor $\mathfrak{Quot}^{\Phi,\mathscr{L}}_{E/X/S}$ is representable by a scheme $\mathrm{Quot}^{\Phi,\mathscr{L}}_{E/X/S}$ which can be embedded over S as a closed subscheme of $\mathbb{P}(F)$ for some vector bundle F on S. Moreover, F can be taken as $\bigwedge^d(W \otimes \mathrm{Sym}^{d'} V)$.*

Remark 5.3. When V and W are trivial, then an immediate corollary of the theorem above is that $\mathrm{Quot}^{\Phi,\mathscr{L}}_{E/X/S}$ can be embedded over S as a closed subscheme of $\mathbb{P}^r_S$ for some $r \geq 0$.

A reduction. First, we can reduce theorem 5.2 to the case where $X = \mathbb{P}(V)$ and $E = \pi^*(W)$ by noticing that

- (i) Tensoring with $\mathscr{L}^{\otimes v}$ gives an isomorphism of functors from $\mathfrak{Quot}^{\Phi,\mathscr{L}}_{E/X/S}$ to $\mathfrak{Quot}^{\Psi,\mathscr{L}}_{E(v)/X/S}$ where $\Psi(\lambda) = \Phi(\lambda + v)$.
- (ii) If $\varphi : E \to G$ is a surjective homomorphism of coherent sheaves on X, then the corresponding natural transformation of functors $\mathfrak{Quot}^{\Phi,\mathscr{L}}_{G/X/S} \to \mathfrak{Quot}^{\Phi,\mathscr{L}}_{E/X/S}$ is a closed embedding. This can be seen easily by applying remark 3.5 to the $\ker\varphi \to F$, where F is any quotient of E.

Use of m-regularity. Now, by lemma 2.2 and theorem 2.3, we know that for each $\Phi \in \mathbb{Q}[\lambda]$, there exists an integer m such that for any $s : \mathrm{Spec}\,k \in S$, where k is any field, and any quotient $E_s \to \mathscr{F}$ with Hilbert polynomial Φ and kernel $\mathscr{K}$, we have the following for $r \geq m$

- (i) $H^0(X_s, E_s(r)), H^0(X_s, \mathscr{F}(r))$ and $H^0(X_s, \mathscr{K}(r))$ are generated by global sections.
- (ii) $H^i(X_s, E_s(r)) = H^i(X_s, \mathscr{F}(r)) = H^i(X_s, \mathscr{K}(r)) = 0, \forall i \geq 1$.

Now, if $(q : E_T \to \mathscr{F}) \in \mathfrak{Quot}^{\Phi,\mathscr{L}}_{E/X/S}(T)$, for some scheme T over S, then by the base change theorems for coherent cohomology we know that for $r \geq m$ (note that $E_T, \mathscr{F}$ are flat over T, which implies the same thing for $\mathscr{K} = \ker q$)

(i) $\pi_{T_*}\mathscr{K}(r), \pi_{T_*}E_T(r)$ and $\pi_{T_*}\mathscr{F}(r)$ are locally free of fixed ranks determined by $\dim V, \dim W, r$ and Φ. Moreover, $\pi_T^*\pi_{T*}(\mathscr{K}(r)) \to \mathscr{K}(r)$, $\pi_T^*\pi_{T*}(E_T(r)) \to E_T(r)$ and $\pi_T^*\pi_{T*}(\mathscr{F}(r)) \to \mathscr{F}(r)$ are surjective.

(ii) $R^i\pi_{T*}\mathscr{K}(r) = R^i\pi_{T*}E_T(r) = R^i\pi_{T*}\mathscr{F}(r)) = 0$ for $i \geq 1$.

Embed into the Grassmannian. The previous section gives us a surjection $\pi_{T*}E_T(r) \to \pi_{T*}\mathscr{F}(r)$, and hence, an element in $\mathfrak{G}rass(\pi_{T*}E_T(r), \Phi(r))(T) = \mathfrak{G}rass(W \otimes \operatorname{Sym}^r V, \Phi(r))(T)$. We have thus constructed a natural transformation

$$\alpha : \mathfrak{Q}uot^{\Phi,\mathscr{L}}_{E/X/S} \to \mathfrak{G}rass(W \otimes_{\mathscr{O}_S} \operatorname{Sym}^r V, \Phi(r)).$$

The following commutative diagram, where the rows are exact and the vertical maps are surjective, allows us to show that α is injective

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_T^*\pi_{T*}\mathscr{K}(r) & \longrightarrow & \pi_T^*\pi_{T*}(E_T(r)) & \longrightarrow & \pi_T^*\pi_{T*}\mathscr{F}(r) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{K}(r) & \longrightarrow & E_T(r) & \longrightarrow & \mathscr{F}(r) & \longrightarrow & 0 \end{array}$$

Indeed, if we let $\mathfrak{Q}$ be the functor defined by (note that there is no flatness requirement)

$$\mathfrak{Q}(T) = \langle \text{Equivalence class of quotients of } E_T \rangle,$$

then taking the quotient of $E_T(r)$ by the image of $\pi_T^*\pi_{T*}\mathscr{K}(r) \to E_T(r)$ gives us a natural transformation of functors $\beta : \mathfrak{G}rass(W \otimes_{\mathscr{O}_S} \operatorname{Sym}^r V, \Phi(r)) \to \mathfrak{Q}$ whose composition

$$\mathfrak{Q}uot^{\Phi,\mathscr{L}}_{E/X/S} \xrightarrow{\alpha} \mathfrak{G}rass(W \otimes_{\mathscr{O}_S} \operatorname{Sym}^r V, \Phi(r)) \xrightarrow{\beta} \mathfrak{Q}$$

is just the natural inclusion of flat quotients into all quotients. Moreover, for any $T \to S$ in Sch_S, $\beta_T^{-1}(\operatorname{Im}(\beta_T \circ \alpha_T)) = \operatorname{Im}(\alpha_T)$. In other words, the image of α is saturated with respect to β.

Representability via flattening stratification. By theorem 4.2 on flattening stratification, $\beta \circ \alpha$ is (relatively) representable by a locally closed embedding. The lemma below, which is almost a tautology, will then conclude that $\mathfrak{Q}uot^{\Phi,\mathscr{L}}_{E/X/S}$ is representable as a locally closed subscheme of $\operatorname{Grass}(W \otimes_{\mathscr{O}_S} \operatorname{Sym}^r V, \Phi(r))$. Using valuative criterion for properness, we can check easily that $\mathfrak{Q}uot^{\Phi,\mathscr{L}}_{E/X/S}$ is proper, and hence, it is actually a closed subscheme of the Grassmannian. Using the Plücker embedding, we see that $\operatorname{Quot}^{\Phi,\mathscr{L}}_{E/X/S}$ is a closed subscheme of $\mathbb{P}(\bigwedge^{\Phi(r)}(W \otimes_{\mathscr{O}_S} \operatorname{Sym}^r V))$. From this construction, we also see that if $\mathscr{F}_Q$ is the universal quotient, then $\det \pi_{\operatorname{Quot}^{\Phi,\mathscr{L}}_{E/X/S}*}\mathscr{F}_Q(r)$ is relatively very ample.

Lemma 5.4. *Let $\alpha : F \to G$ and $\beta : G \to H$ be morphisms of contravariant functors from* Sch_S *to* Sets. *Suppose α is injective, G and $\beta \circ \alpha$ are representable, and moreover, the image of α is saturated with respect to β, then $F \cong G \times_H F$, which is representable.*

Remark 5.5. We note that the version by Altman and Kleiman that we just mentioned above is not weaker than that of Grothendieck. This is because with the stronger hypothesis, they can prove a stronger form of projectivity for the Quot schemes.

Remark 5.6. For the version of Grothendieck, the proof above still implies that $\mathfrak{Q}uot^{\Phi,\mathscr{L}}_{E/X/S}$ is representable by a proper scheme over S. This is because representability is Zariski local on the base, and locally, E is a quotient of a vector bundle of the form $\mathscr{O}_X(-n)^q$. But now, note that while projectivity cannot be checked Zariski locally on the base, the notion of relatively very ample can be. Locally, as above, we see that if $\mathscr{F}_Q$ is the universal quotient, then $\det \pi_{\operatorname{Quot}^{\Phi,\mathscr{L}}_{E/X/S}*}\mathscr{F}_Q(r)$ is relatively very ample over an affine open subscheme of S, when r is big enough. Since S is Noetherian, we can take an r big enough globally, and we get a relative very ample line bundle $\det \pi_{\mathrm{Quot}^{\Phi,\mathscr{L}}_{E/X/S}*}\mathscr{F}_Q(r)$.

Remark 5.7. In theorem 5.1, we can replace projectivity by quasi-projectivity. The resulting Quot scheme will then be quasi projective instead of projective. To see this, we only need to embed X into its scheme theoretic closure $\overline{X}$ in $\mathbb{P}(\mathscr{F})$, and also extend E to a coherent $\mathscr{O}_{\overline{X}}$-module $\overline{E}$. Theorem 5.1 then asserts the representability of $\mathfrak{Quot}^{\Phi,\mathscr{L}}_{\overline{E}/\overline{X}/S}$. The following lemma allows us to construct $\mathrm{Quot}^{\Phi,\mathscr{L}}_{E/X/S}$ as an open subscheme of $\mathrm{Quot}^{\Phi,\mathscr{L}}_{\overline{E}/\overline{X}/S}$.

Lemma 5.8. *Let $\pi : X \to S$ be a proper morphism of noetherian schemes. Let $Z \subset X$ be a closed subscheme, and $\mathscr{F}$ a coherent sheaf on X. Then there exists an open subscheme $S' \subset S$ with the universal property that a morphism $T \to S$ factors through S' if and only if the support of the pull-back $\mathscr{F}_T$ on $X_T = X \times_S T$ is disjoint from $Z_T = Z \times_S T$.*

Proof (Sketch). We immediately reduce to the case $X = Z$, and ask for a universal open subscheme S' of S such that $\mathscr{F}|_{\pi^{-1}(S')} = 0$. We see at once that $S' = S \setminus \pi(\mathrm{Supp}\mathscr{F})$ is the desired open subscheme. □

6. An Application

In this section, we will sketch the construction of the Hom-scheme.

Definition 6.1 (The Hom functor)**.** *Let X and Y be schemes over a base S, let $\mathfrak{Hom}_S(X,Y)$ be the contravariant functors from* Sch_S *to* Sets *defined by*

$$\mathfrak{Hom}_S(X,Y)(T) = \mathrm{Hom}_T(X_T, Y_T),$$

where T is any scheme in Sch_S.

Definition 6.2 (The Hom scheme)**.** *Let X, Y and S as above, then $\mathrm{Hom}_S(X,Y)$ is the scheme representing the functor $\mathfrak{Hom}_S(X,Y)$ (if it exists).*

Theorem 6.3. *Let S be a noetherian scheme, X a projective scheme over S, and Y a quasi-projective scheme over S. Assume moreover that X is flat over S. Then $\mathfrak{Hom}_S(X,Y)$ is representable by an open subscheme $\mathrm{Hom}_S(X,Y)$ of $\mathrm{Hilb}_{X\times_S Y/S}$.*

Proof (Sketch). Since Y is separated, for each $f \in \mathfrak{Hom}_S(X,Y)(T)$, the graph Γ_f is a closed embedding $X_T \to X_T \times_T Y_T = (X \times_S Y)_T$. Since X is proper and flat over S, Γ_f is proper and flat over T. Thus, we get a natural transformation of functors, which is injective

$$\begin{aligned} c : \mathfrak{Hom}_S(X,Y) &\to \mathfrak{Hilb}_{X\times_S Y/S} \\ f &\mapsto \Gamma_f. \end{aligned}$$

Note also that $\Gamma \in \mathfrak{Hilb}_{X\times_S Y/S}(T)$ is in the image of c_T if and only if the composition $\Gamma \to X_T \times_T Y_T \to X_T$ is an isomorphism. In this case, the map f can be recovered by taking the inverse of this isomorphism and project to the Y factor: $X_T \to \Gamma \to X_T \times_T Y_T \to Y_T$.

From the above, we already know that $\mathfrak{Hilb}_{X\times_S Y/S}$ is representable by $\mathrm{Hilb}_{X\times_S Y/S}$. To conclude, we need to show that c is representable by an open immersion. This is the content of the theorem 6.4 below. □

Theorem 6.4. *Let S be a noetherian scheme, and let $f : X \to S$ and $g : Y \to S$ be proper flat morphisms. Let $\pi : X \to Y$ be any projective morphism, compatible with f and g. Then S has open subscheme $S_2 \subset S_1 \subset S$ that satisfy the following universal property:*

(i) For any T in Sch_S, $\pi_T : X_T \to Y_T$ is a flat morphism if and only if the structure morphism $T \to S$ factors through S_1. (This part does not require π to be projective.)
(ii) For any T in Sch_S, $\pi_T : X_T \to Y_T$ is an isomorphism if and only if the structure morphism $T \to S$ factors through S_2.

Proof (Sketch).

(i) This is a consequence of the openness of a flat morphism, and the local criterion for flatness. The proof is in a similar style as that of lemma 5.8.

(ii) Let $\pi_1 : X_1 \to Y_1$ be the pull back of π over S_1. Let $\mathscr{L}$ be a relatively very ample line bundle for π_1. Using noetherian-ness and base change theorems (note that X_1 is flat over Y_1), we can find an m such that $R^i\pi_{1*}\mathscr{L}^m = 0$ for all $i \geq 1$ and $\pi_{1*}\mathscr{L}$ is a locally free sheaf on Y_1. Let $U \subset Y_1$ be the open subscheme such that $\pi_{1*}\mathscr{L}^m$ has rank 1. Note that then, $\pi_1^{-1}(U) \to U$ is an isomorphism. But since we are working relatively over S_1, we need to shrink this further; namely, we take $S_2 = S_1 - f(Y_1 - U)$. The pull back of π_1 over S_2, $\pi_2 : X_2 \to Y_2$, is then an isomorphism. The universal property is checked by an application of the base change theorems since all we need to check is that S_2 is the universal subscheme such that $\pi_{2*}\mathscr{L}^m$ is an invertible sheaf.

$\square$

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