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# Configuration spaces as commutative monoids

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*With an appendix with Quoc P. Ho*[1]

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**Abstract**

After one-point compactification, the collection of all unordered configuration spaces of a manifold admits a commutative multiplication by superposition of configurations. We explain a simple (derived) presentation for this commutative monoid object. Using this presentation, one can quickly deduce Knudsen's formula for the rational cohomology of configuration spaces, prove rational homological stability and understand how automorphisms of the manifold act on the cohomology of configuration spaces. Similar considerations reproduce the work of Farb–Wolfson–Wood on homological densities.

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## 1 | INTRODUCTION

Let $M$ be the interior of a connected compact manifold with boundary. The one-point compactification of the space $C_n(M)$ of unordered configurations in $M$ may be written as

$$C_n(M)^+ = \left[\frac{(M^+)^{\wedge n}}{\text{locus where two points coincide}}\right]_{\mathfrak{S}_n}, \tag{1.1}$$

the quotient formed in pointed spaces. Not-necessarily-disjoint union of unordered configurations defines a *superposition product*

$$C_n(M)^+ \wedge C_{n'}(M)^+ \longrightarrow C_{n+n'}(M)^+,$$

which is associative, commutative and unital. This gives a unital commutative monoid object in the symmetric monoidal category $\mathsf{Top}_*^{\mathbb{N}}$ of $\mathbb{N}$-graded pointed spaces:

$$\mathbf{C}(M) : n \longmapsto C_n(M)^+.$$

The goal of this note is to explain and exploit this algebraic structure.

In the following, for a pointed space $X$, we write $X[n]$ for the $\mathbb{N}$-graded pointed space which consists of $X$ in grading $n$ and the point in all other gradings, and write $\mathbf{Com}(Y)$ for the free unital commutative monoid on an object $Y \in \mathsf{Top}_*^{\mathbb{N}}$.

**Theorem 1.1.** *There is a pushout square*

$$\begin{array}{ccc} \mathbf{Com}(M^+[2]) & \xrightarrow{\epsilon} & S^0[0] \\ \downarrow{\scriptstyle\Delta} & & \downarrow \\ \mathbf{Com}(M^+[1]) & \longrightarrow & \mathbf{C}(M) \end{array}$$

*of unital commutative monoids in* $\mathsf{Top}_*^{\mathbb{N}}$, *where* $\epsilon$ *is the augmentation and* $\Delta$ *is induced by the diagonal inclusion* $M^+ \to [M^+ \wedge M^+]_{\mathfrak{S}_2} = \mathbf{Com}(M^+[1])(2)$. *Furthermore, this square is a homotopy pushout, that is, there is an induced equivalence*

$$\mathbf{Com}(M^+[1]) \otimes^{\mathbb{L}}_{\mathbf{Com}(M^+[2])} S^0[0] \xrightarrow{\sim} \mathbf{C}(M).$$

That the square is a strict pushout of unital commutative monoids is elementary: it means identifying $\mathbf{C}(M)$ as the quotient of the based symmetric power monoid of $M^+$ by the ideal given by those tuples which contain a repeated element, which is a reformulation of (1.1). The content of the theorem is that the square is also a homotopy pushout, rendering it amenable to homological calculation.

More generally, let $\pi : L \to M$ be a vector bundle, and let

$$C_n(M; L)^+ = \left[ \frac{(L^+)^{\wedge n}}{\text{locus where two points have the same projection in } M} \right]_{\mathfrak{S}_n}.$$

These assemble in the same way to a unital commutative monoid object $\mathbf{C}(M; L)$. (Of course, more general spaces of labels can be implemented too, but the above suffices for us.)

**Theorem 1.2.** *There is a pushout square*

$$\begin{array}{ccc} \mathbf{Com}([(L \oplus L)^+]_{\mathfrak{S}_2}[2]) & \xrightarrow{\epsilon} & S^0[0] \\ \downarrow{\scriptstyle\Delta} & & \downarrow \\ \mathbf{Com}(L^+[1]) & \longrightarrow & \mathbf{C}(M; L) \end{array}$$

*of unital commutative monoids in* $\mathsf{Top}_*^{\mathbb{N}}$, *where* $\epsilon$ *is the augmentation and* $\Delta$ *is induced by the diagonal inclusion* $[(L \oplus L)^+]_{\mathfrak{S}_2} \to [L^+ \wedge L^+]_{\mathfrak{S}_2} = \mathbf{Com}(L^+[1])(2)$. *Furthermore, this square is a homotopy pushout, that is, there is an induced equivalence*

$$\mathbf{Com}(L^+[1]) \otimes^{\mathbb{L}}_{\mathbf{Com}([(L \oplus L)^+]_{\mathfrak{S}_2}[2])} S^0[0] \xrightarrow{\sim} \mathbf{C}(M; L).$$

This strictly generalises Theorem 1.1, which is the case where $L$ is the zero-dimensional vector bundle, so we shall mostly focus on Theorem 1.2 for the rest of the paper.

Recall that the derived relative tensor product may be computed by the two-sided bar construction, formed in $\mathsf{Top}_*^{\mathbb{N}}$, so the conclusion of Theorem 1.2 can equivalently be stated as an equivalence

$$B(\mathbf{Com}(L^+[1]), \mathbf{Com}([(L \oplus L)^+]_{\mathfrak{S}_2}[2]), S^0[0]) \xrightarrow{\sim} \mathbf{C}(M; L). \tag{1.2}$$

This formula has many applications to the homology of configuration spaces. As one application, we will show how to recover Knudsen's [18] formula for $H^*(C_n(M); \mathbb{Q})$ in terms of the compactly supported $\mathbb{Q}$-cohomology of $M$ and its cup-product map, which, in particular, quickly implies homological stability. As another application, we will show that the action on $H^*(C_n(M); \mathbb{Q})$ of the group of proper homotopy self-equivalences of $M$ factors over a surprisingly small group. Finally, in Appendix A, written with Quoc P. Ho, we show how similar considerations reproduce the work of Farb–Wolfson–Wood [10] on homological densities.

**Context**. This note is my attempt to give a topological implementation of some of the sheaf-theoretic ideas of Banerjee [3] in the case of configuration spaces. The applications to the homology of configuration spaces given in Section 2 arise by taking singular chains of the equivalence (1.2) to obtain a derived tensor product description of the chains on $\mathbf{C}(M; L)$: this description will also follow from [2] as explained in [3, Remark 1.1]. As such, the purpose of this paper is

(i) to give a space-level implementation/interpretation of Banerjee's ideas in a specific case, in order to popularise them among topologists, and
(ii) to explain how several classical, recent and new results about the rational homology of configuration spaces can be obtained very efficiently from (1.2) (or its chain-level analogue).

Everything I will describe has much to do with the work of Ho [13, 14], Petersen [21], Knudsen [18], Getzler [11, 12], Kallel [15], Bödigheimer–Cohen–Milgram [4], Segal [23] and Arnol'd [1].

## 2 | APPLICATIONS

### 2.1 | Homology of configuration spaces

Let $M$ be $d$-dimensional. The space $C_n(M; L)^+$ is the one-point compactification of the $n \cdot (d + \dim(L))$-dimensional manifold

$$C_n(M; L) := \left[L^n \setminus \{(l_1, \ldots, l_n) \mid \pi(l_i) = \pi(l_j) \text{ and } i \neq j\}\right]_{\mathfrak{S}_n}.$$

This is a vector bundle over $C_n(M)$, but is a manifold itself and is orientable if and only if the manifold $L$ is orientable and even-dimensional. To arrange this, we can take the vector bundle $W$ given by the orientation line of $M$ plus $(d-1)$ trivial line bundles. Thus, by Poincaré duality, we have

$$H^*(C_n(M); \Bbbk) \cong H^*(C_n(M; W); \Bbbk) \cong \widetilde{H}_{2dn-*}(C_n(M; W)^+; \Bbbk).$$

In view of this, the bar construction description (1.2) can be used, in combination with the homology of free commutative monoids (see [20]), to investigate $H^*(C_n(M);\Bbbk)$. We do not pursue this in general here, but rather focus on the case $\Bbbk = \mathbb{Q}$, where a complete answer is possible, and reproduces a formula of Knudsen.

## 2.2 | Revisiting Knudsen's formula

For an $\mathbb{N}$-graded pointed space, we write $H_{n,i}(X) := \widetilde{H}_i(X(n))$, and similarly for chains. Write $S^*(V)$ for the free graded commutative algebra on a homologically graded vector space $V$, that is, $S^*(V) = \bigoplus_{n\geqslant 0}[V^{\otimes n}]_{\mathfrak{S}_n}$, where the Koszul sign rule is implemented. If $V$ is equipped with additional $\mathbb{N}$-grading, then this is inherited by $S^*(V)$ (but there is no Koszul sign rule associated to the $\mathbb{N}$-grading, only to the homological grading).

We consider $\mathbf{C}(M;W)$. There is a map $\widetilde{C}_*(W^+;\mathbb{Q})[1] \to C_{*,*}(\mathbf{Com}(W^+[1]);\mathbb{Q})$ and, using the Eilenberg–Zilber maps, it extends to a map of cdga's

$$S^*\left(\widetilde{C}_*(W^+;\mathbb{Q})[1]\right) \longrightarrow C_{*,*}(\mathbf{Com}(W^+[1]);\mathbb{Q}),$$

which is an equivalence (since the maps $[(W^+)^{\wedge n}]_{h\mathfrak{S}_n} \to [(W^+)^{\wedge n}]_{\mathfrak{S}_n}$ are rational homology isomorphisms). Similarly, there is an equivalence of cdga's

$$S^*\left(\widetilde{C}_*([(W\oplus W)^+]_{\mathfrak{S}_2};\mathbb{Q})[2]\right) \longrightarrow C_{*,*}(\mathbf{Com}([(W\oplus W)^+]_{\mathfrak{S}_2}[2]);\mathbb{Q}).$$

Furthermore, one may choose formality equivalences

$$\widetilde{H}_*(W^+;\mathbb{Q}) \longrightarrow \widetilde{C}_*(W^+;\mathbb{Q})$$
$$\widetilde{H}_*([(W\oplus W)^+]_{\mathfrak{S}_2};\mathbb{Q}) \longrightarrow \widetilde{C}_*([(W\oplus W)^+]_{\mathfrak{S}_2};\mathbb{Q}),$$

that is, chain maps inducing the identity on homology, and hence obtain equivalences

$$S^*\left(\widetilde{H}_*(W^+;\mathbb{Q})[1]\right) \longrightarrow S^*\left(\widetilde{C}_*(W^+;\mathbb{Q})[1]\right)$$
$$S^*\left(\widetilde{H}_*([(W\oplus W)^+]_{\mathfrak{S}_2};\mathbb{Q})[2]\right) \longrightarrow S^*\left(\widetilde{C}_*([(W\oplus W)^+]_{\mathfrak{S}_2};\mathbb{Q})[2]\right)$$

of cdga's. In $\mathbb{N}$-grading 2, the map $\Delta$ induces a map

$$\delta_* : \widetilde{H}_*([(W\oplus W)^+]_{\mathfrak{S}_2};\mathbb{Q}) \xrightarrow{\Delta_*} \widetilde{H}_*([(W^+)^{\wedge 2}]_{\mathfrak{S}_2};\mathbb{Q}) \cong \left[\widetilde{H}_*(W^+;\mathbb{Q})^{\otimes 2}\right]_{\mathfrak{S}_2}.$$

With these choices, the square

$$\begin{array}{ccc} S^*\left(\widetilde{H}_*([(W\oplus W)^+]_{\mathfrak{S}_2};\mathbb{Q})[2]\right) & \xrightarrow{\sim} & C_{*,*}(\mathbf{Com}([(W\oplus W)^+]_{\mathfrak{S}_2}[2]);\mathbb{Q}) \\ \downarrow{\scriptstyle S^*(\delta_*)} & & \downarrow{\scriptstyle \Delta_\#} \\ S^*\left(\widetilde{H}_*(W^+;\mathbb{Q})[1]\right) & \xrightarrow{\sim} & C_{*,*}(\mathbf{Com}(W^+[1]);\mathbb{Q}) \end{array}$$

need not commute, but does commute up to homotopy in the category of cdga's because the two chain maps $\widetilde{H}_*([(W\oplus W)^+]_{\mathfrak{S}_2};\mathbb{Q}) \to \widetilde{C}_*([(W^+)^{\wedge 2}]_{\mathfrak{S}_2};\mathbb{Q})$ induce the same map on homology, namely $\delta_*$, so are chain homotopic. The bar construction description then gives an identification

$$\mathrm{Tor}_*^{S^*(\widetilde{H}_*([(W\oplus W)^+]_{\mathfrak{S}_2};\mathbb{Q})[2])}(S^*\left(\widetilde{H}_*(W^+;\mathbb{Q})[1]\right),\mathbb{Q}[0]) \cong H_{*,*}(\mathbf{C}(M;W);\mathbb{Q}).$$

Recall that for a free graded-commutative algebra $S^*(V)$ on a homologically graded vector space $V$ (perhaps equipped with a further $\mathbb{N}$-grading), there is a free resolution of the trivial left $S^*(V)$-module $\mathbb{Q}$ given by $\epsilon : S^*(V \oplus \Sigma V) \xrightarrow{\sim} \mathbb{Q}$ equipped with the differential given by $\partial(\Sigma v) = v$ and extended by the Leibniz rule. It is usually called the Koszul resolution. It is indeed a resolution because it is the free graded-commutative algebra on the acyclic chain complex $\Sigma V \xrightarrow{id} V$, and over $\mathbb{Q}$ taking homology commutes with the formation of symmetric powers. Applying this resolution to calculate the Tor groups above gives the complex

$$\left(S^*\left(\widetilde{H}_*(W^+;\mathbb{Q})[1] \oplus \Sigma\widetilde{H}_*([(W\oplus W)^+]_{\mathfrak{S}_2};\mathbb{Q})[2]\right),\partial\right)$$

with differential given by $\partial(\Sigma x) = \Delta_*(x) \in S^2(\widetilde{H}_*(W^+[1];\mathbb{Q}))$ for $x \in \widetilde{H}_*([(W\oplus W)^+]_{\mathfrak{S}_2};\mathbb{Q})[2]$, and extended by the Leibniz rule. This can be simplified as follows. If $M$ is $d$-dimensional, then the Thom isomorphism gives $\widetilde{H}_*(W^+;\mathbb{Q}) = \Sigma^d\widetilde{H}_*(M^+;\mathbb{Q}^{w_1})$, where $\mathbb{Q}^{w_1}$ is the orientation local system of $M$. It also gives $\widetilde{H}_*((W\oplus W)^+;\mathbb{Q}) = \Sigma^{2d}\widetilde{H}_*(M^+;\mathbb{Q})$. The involution swapping the two $W$ factors acts as $(-1)^d$ on the Thom class, so because the map $[(W\oplus W)^+]_{h\mathfrak{S}_2} \to [(W\oplus W)^+]_{\mathfrak{S}_2}$ is a rational equivalence, we find

$$\widetilde{H}_*([(W\oplus W)^+]_{\mathfrak{S}_2};\mathbb{Q}) = \begin{cases}\Sigma^{2d}\widetilde{H}_*(M^+;\mathbb{Q}) & d \text{ even}\\ 0 & d \text{ odd.}\end{cases}$$

This lets us write the complex as

$$\left(S^*\left(\Sigma^d\widetilde{H}_*(M^+;\mathbb{Q}^{w_1})[1] \oplus \begin{cases}\Sigma^{2d+1}\widetilde{H}_*(M^+;\mathbb{Q}) & d \text{ even}\\ 0 & d \text{ odd}\end{cases}[2]\right),\partial\right), \tag{2.1}$$

where the differential is dual to the map $S^2(H_c^*(M;\mathbb{Q}^{w_1})) \to H_c^*(M;\mathbb{Q})$ induced by cup product, so following Knudsen, we can recognise this complex as the Chevalley–Eilenberg complex for the bigraded Lie algebra $H_c^*(M;\mathrm{Lie}(\Sigma^{d-1}\mathbb{Q}^{w_1}[1]))$. Thus,

$$H^{2nd-*}(C_n(M);\mathbb{Q}) \cong \widetilde{H}_*(C_n(M;W)^+;\mathbb{Q}) \cong H_*^{\mathrm{Lie}}(H_c^*(M;\mathrm{Lie}(\Sigma^{d-1}\mathbb{Q}^{w_1}[1])))(n).$$

After appropriate dualisations and reindexings, this agrees with Knudsen's formula.

## 2.3 | Homological stability

Stability for the homology of configuration spaces is by now a classical subject, with a large number of contributions by many authors: notable examples are [1, 6, 8, 9, 16, 18, 22, 23]. In particular, Knudsen has explained [18, Section 5.3] how his formula implies rational (co)homological stability for the spaces $C_n(M)$. Let us briefly review this from the point of view taken here.

There is a canonical element $[M] \in \widetilde{H}_d(M^+;\mathbb{Q}^{w_1}) \cong \widetilde{H}_{2d}(W^+;\mathbb{Q})$, and choosing a cycle representing this element provides a map

$$\sigma : \Sigma^{2d}\mathbb{Q}[1] \longrightarrow C_{*,*}(\mathbf{Com}(W^+[1]);\mathbb{Q}) \longrightarrow C_{*,*}(\mathbf{C}(M;W);\mathbb{Q}).$$

Multiplication by this element defines a map

$$(\sigma \cdot -)_* : \widetilde{H}_{n-1,2d(n-1)-i}(\mathbf{C}(M;W);\mathbb{Q}) \longrightarrow \widetilde{H}_{n,2dn-i}(\mathbf{C}(M;W);\mathbb{Q}),$$

which under Poincaré duality gives a map $H^i(C_{n-1}(M);\mathbb{Q}) \to H^i(C_n(M);\mathbb{Q})$; this can be checked to be the transfer map that sums over all ways of forgetting one of the $n$ points, see [18, Section 5.2] [25, Section 2.6].

Writing $C_{*,*}(\mathbf{C}(M;W);\mathbb{Q})/\sigma$ for the mapping cone of left multiplication by $\sigma$, the discussion above shows that its homology is calculated by a complex

$$\left(S^*\left(\Sigma^d\frac{\widetilde{H}_*(M^+;\mathbb{Q}^{w_1})}{\langle[M]\rangle}[1] \oplus \begin{cases}\Sigma^{2d+1}\widetilde{H}_*(M^+;\mathbb{Q}) & d \text{ even}\\ 0 & d \text{ odd}\end{cases}[2]\right), \partial\right).$$

As $M$ is connected, if we assume that $d \geqslant 3$, then the bigraded vector spaces $\Sigma^d\frac{\widetilde{H}_*(M^+;\mathbb{Q}^{w_1})}{\langle[M]\rangle}[1]$ and $\Sigma^{2d+1}\widetilde{H}_*(M^+;\mathbb{Q})[2]$ both vanish in bidegrees $(n,j)$ satisfying $j > (2d-1)n$, and hence so does the free graded commutative algebra on them. This translates to $H^i(C_{n-1}(M);\mathbb{Q}) \to H^i(C_n(M);\mathbb{Q})$ being surjective for $i < n$ and an isomorphism for $i < n-1$. For $d = 2$, the same considerations give surjectivity for $i < \frac{1}{2}n$, and so on (a more careful analysis gives a slope 1 range in this case too, see [18, Proof of Theorem 1.3]).

Analysing the complex (2.1) can also establish other kinds of stability results, for example, [7, 17, 26].

## 2.4 | The action of automorphisms on unordered configurations

Using Knudsen's formula, it is possible to mislead yourself into thinking that homeomorphisms of $M$ (or indeed pointed homotopy self-equivalences of $M^+$) act on $H_*(C_n(M);\mathbb{Q})$ via their action on $H_*(M;\mathbb{Q})$: in other words, that such maps which act trivially on the homology of $M$ also act trivially on the homology of $C_n(M)$. This is not true: in the case of surfaces, see Bianchi [5, Section 7], Looijenga [19] and the complete analysis given by Stavrou [24].

From the point of view taken here, this phenomenon can be explained as follows. For simplicity, suppose that $M$ is orientable, and first suppose that it is odd-dimensional. Then, $H^*(C_n(M);\mathbb{Q}) \cong \widetilde{H}_{2dn-*}(C_n(M;M\times\mathbb{R}^d)^+;\mathbb{Q})$ and the analysis of Section 2.2 applied to $\mathbf{C}(M;M\times\mathbb{R}^d)$ shows that $\mathbf{Com}(S^d \wedge M^+[1]) \to \mathbf{C}(M;M\times\mathbb{R}^d)$ is a rational homology isomorphism. So we find:

**Theorem 2.1.** *If $M$ is orientable and odd-dimensional, then a pointed homotopy self-equivalence of $M^+$ which acts trivially on $\widetilde{H}_*(M^+;\mathbb{Q})$ also acts trivially on $H^*(C_n(M);\mathbb{Q})$.*

The even-dimensional case is more interesting. As $M$ is assumed orientable, in this case, the twisting by $W$ can be dispensed with. It is technically convenient here—for reasons of symmetric monoidality—to work in the category of simplicial $\mathbb{Q}$-modules rather than chain complexes. We write $- \odot -$ for the tensoring of this category over simplicial sets. For a space $X$, let us abbreviate $\mathbb{Q}[X] := \mathbb{Q}[\mathrm{Sing}_\bullet(X)]$, and if it is based, then let $\widetilde{\mathbb{Q}}[X] = \mathbb{Q}[X]/\mathbb{Q}[*]$. The discussion in the previous section, ignoring the formality step and translated to simplicial $\mathbb{Q}$-modules, shows that given the simplicial module $\widetilde{\mathbb{Q}}[M^+]$ and the map $\delta : \widetilde{\mathbb{Q}}[M^+] \to \left[\widetilde{\mathbb{Q}}[M^+]^{\otimes 2}\right]_{\mathfrak{S}_2}$ induced by the diagonal $M^+ \to M^+ \wedge M^+$, we may form the two-sided bar construction

$$B\big(S^*\big(\widetilde{\mathbb{Q}}[M^+][1]\big), S^*\big(\widetilde{\mathbb{Q}}[M^+][2]\big), \mathbb{Q}[0]\big) \tag{2.2}$$

whose bigraded homotopy groups are identified with $\widetilde{H}_*(C_*(M)^+; \mathbb{Q})$.

A homeomorphism of $M$, or a pointed homotopy self-equivalence of $M^+$, induces an equivalence $\phi : \widetilde{\mathbb{Q}}[M^+] \to \widetilde{\mathbb{Q}}[M^+]$ such that $\delta \circ \phi = ([\phi \otimes \phi]_{\mathfrak{S}_2}) \circ \delta$, meaning that the diagram of simplicial commutative rings

$$\begin{array}{ccccc}
S^*\left(\widetilde{\mathbb{Q}}[M^+][1]\right) & \xleftarrow{S^*(\delta)} & S^*\left(\widetilde{\mathbb{Q}}[M^+][2]\right) & \xrightarrow{\epsilon} & \mathbb{Q}[0] \\
\big\downarrow{\scriptstyle S^*(\phi)} & & \big\downarrow{\scriptstyle S^*(\phi)} & & \big\downarrow{\scriptstyle id} \\
S^*\left(\widetilde{\mathbb{Q}}[M^+][1]\right) & \xleftarrow{S^*(\delta)} & S^*\left(\widetilde{\mathbb{Q}}[M^+][2]\right) & \xrightarrow{\epsilon} & \mathbb{Q}[0]
\end{array}$$

is commutative, which induces a self-equivalence of the two-sided bar construction (2.2). This corresponds to the induced action on $\widetilde{H}_*(C_*(M)^+; \mathbb{Q})$.

However, a weaker kind of data suffices to get an induced equivalence on two-sided bar constructions. An equivalence $\phi : \widetilde{\mathbb{Q}}[M^+] \to \widetilde{\mathbb{Q}}[M^+]$ together with a homotopy $h : \delta \circ \phi \Rightarrow ([\phi \otimes \phi]_{\mathfrak{S}_2}) \circ \delta$ gives a diagram of simplicial commutative rings as above where the right-hand square commutes and the left-hand squares commutes up to the homotopy $S^*(h) : S^*(\Delta^1 \odot \widetilde{\mathbb{Q}}[M^+][2]) \to S^*(\widetilde{\mathbb{Q}}[M^+][1])$. This data suffice to obtain a self-equivalence $\chi(\phi, h)$ of the two-sided bar construction (2.2), as the zig-zag

$$\begin{array}{ccc}
B\left(S^*\left(\widetilde{\mathbb{Q}}[M^+][1]\right), S^*\left(\Delta^1 \odot \widetilde{\mathbb{Q}}[M^+][2]\right), \mathbb{Q}[0]\right) & \xleftarrow[B(id, S^*(d_0), id)]{} & B\left(S^*\left(\widetilde{\mathbb{Q}}[M^+][1]\right)', S^*\left(\widetilde{\mathbb{Q}}[M^+][2]\right), \mathbb{Q}[0]\right) \\
\big\uparrow{\scriptstyle B(S^*(\phi), S^*(d_1), id)} & & \big\downarrow{\scriptstyle B(id, S^*(\phi), id)} \\
B\left(S^*\left(\widetilde{\mathbb{Q}}[M^+][1]\right), S^*\left(\widetilde{\mathbb{Q}}[M^+][2]\right), \mathbb{Q}[0]\right) & \xrightarrow{\chi(\phi,h)} & B\left(S^*\left(\widetilde{\mathbb{Q}}[M^+][1]\right), S^*\left(\widetilde{\mathbb{Q}}[M^+][2]\right), \mathbb{Q}[0]\right)
\end{array}$$

where $S^*(\widetilde{\mathbb{Q}}[M^+][1])'$ denotes $S^*(\widetilde{\mathbb{Q}}[M^+][1])$ considered as a $S^*(\widetilde{\mathbb{Q}}[M^+][2])$-module via $S^*(\phi) \circ S^*(\delta)$.

Let $(\phi', h')$ be another such datum, and suppose that there is a homotopy $\Phi : \phi \Rightarrow \phi'$ such that the 2-cell

$$\begin{array}{ccccccc}
\left[\widetilde{\mathbb{Q}}[M^+]^{\otimes 2}\right]_{\mathfrak{S}_2} & & & \xleftarrow{\quad\delta\quad} & & & \widetilde{\mathbb{Q}}[M^+] \\
 & \diagdown\!\!\diagdown & & Id & & \diagup\!\!\diagup & \\
 & & \left[\widetilde{\mathbb{Q}}[M^+]^{\otimes 2}\right]_{\mathfrak{S}_2} & \xleftarrow{\delta} & \widetilde{\mathbb{Q}}[M^+] & & \\
\big\downarrow{\scriptstyle [(\phi')^{\otimes 2}]_{\mathfrak{S}_2}} & [\Phi^{\otimes 2}]_{\mathfrak{S}_2} & \big\downarrow{\scriptstyle [\phi^{\otimes 2}]_{\mathfrak{S}_2}} & h & \big\downarrow{\scriptstyle \phi} & \Phi & \big\downarrow{\scriptstyle \phi'} \\
 & & \left[\widetilde{\mathbb{Q}}[M^+]^{\otimes 2}\right]_{\mathfrak{S}_2} & \xleftarrow[\delta]{} & \widetilde{\mathbb{Q}}[M^+] & & \\
 & \diagup\!\!\diagup & & Id & & \diagdown\!\!\diagdown & \\
\left[\widetilde{\mathbb{Q}}[M^+]^{\otimes 2}\right]_{\mathfrak{S}_2} & & & \xleftarrow[\quad\delta\quad]{} & & & \widetilde{\mathbb{Q}}[M^+]
\end{array} \tag{2.3}$$

is homotopic to $h'$. Then one may check that $\chi(\phi', h')$ is homotopic to $\chi(\phi, h)$. If we let $\Gamma$ denote the set of $(\phi, h)$'s modulo the equivalence relation $(\phi, h) \sim (\phi', h')$ when there exists a homotopy $\Phi$ having the above property, then composition of maps and pasting of homotopies makes $\Gamma$ into a group, which acts on the two-sided bar construction (2.2) in the homotopy category of simplicial $\mathbb{Q}$-modules (and so also acts on its homotopy groups). A pointed homotopy self-equivalence of $M^+$ acts on the two-sided bar construction through $\Gamma$, via elements of the special form $[(\phi, Id)]$.

We may analyse the group $\Gamma$ as follows. There is a homomorphism

$$\begin{aligned} \rho : \Gamma &\longrightarrow \mathrm{Aut}\big(\widetilde{\mathbb{Q}}[M^+]\big) \\ [(\phi, h)] &\longmapsto [\phi] \end{aligned}$$

to the group of homotopy classes of homotopy self-equivalences of $\widetilde{\mathbb{Q}}[M^+]$. Using the Dold–Kan theorem, the latter can be identified with the group of homotopy classes of homotopy self-equivalences of $\widetilde{C}_*(M^+;\mathbb{Q})$, and using a formality equivalence $\widetilde{H}_*(M^+;\mathbb{Q}) \xrightarrow{\sim} \widetilde{C}_*(M^+;\mathbb{Q})$, this is identified with the group $\mathrm{Aut}(\widetilde{H}_*(M^+;\mathbb{Q}))$ of automorphisms of the graded vector space $\widetilde{H}_*(M^+;\mathbb{Q})$. Such an automorphism is in the image of $\rho$ precisely when it preserves the map $\delta_* : \widetilde{H}_*(M^+;\mathbb{Q}) \to [\widetilde{H}_*(M^+;\mathbb{Q})^{\otimes 2}]_{\mathfrak{S}_2}$. The kernel of $\rho$ consists of those $[(\phi, h)]$ such that $\phi$ is homotopic to the identity: by definition of the equivalence relation $\sim$, such an element may be written as $[(Id, h')]$ where $h'$ is obtained from $h$ and a homotopy $\Phi : \phi \Rightarrow id$ by the 2-cell diagram (2.3). Such an $h'$ is a self-homotopy of the map $\delta$, so an element of $\pi_1(\mathrm{map}(\widetilde{\mathbb{Q}}[M^+], \big[\widetilde{\mathbb{Q}}[M^+]^{\otimes 2}\big]_{\mathfrak{S}_2})); \delta)$. The ambiguity in $h'$ when representing $[(\phi, h)] \in \mathrm{Ker}(\rho)$ as $[(Id, h')]$ comes from the choice of the homotopy $\Phi$, so $h'$ is well-defined modulo the ambiguity coming from the self-homotopies $\pi_1(\mathrm{map}(\widetilde{\mathbb{Q}}[M^+], \widetilde{\mathbb{Q}}[M^+]); id)$ of the identity map. In conclusion, this discussion establishes an exact sequence

$$\pi_1\left(\mathrm{map}\left(\widetilde{\mathbb{Q}}[M^+], \widetilde{\mathbb{Q}}[M^+]\right); id\right) \xrightarrow{\delta \circ -} \pi_1\left(\mathrm{map}\left(\widetilde{\mathbb{Q}}[M^+], \big[\widetilde{\mathbb{Q}}[M^+]^{\otimes 2}\big]_{\mathfrak{S}_2}\right); \delta\right) \longrightarrow \Gamma \xrightarrow{\rho} \mathrm{Aut}\left(\widetilde{H}_*(M^+;\mathbb{Q}), \delta_*\right) \longrightarrow 1.$$

Using the Dold–Kan theorem and a formality equivalence again, we can identify the first map in this sequence with

$$\delta_* \circ - : \mathrm{Hom}\big(\Sigma\widetilde{H}_*(M^+;\mathbb{Q}), \widetilde{H}_*(M^+;\mathbb{Q})\big) \longrightarrow \mathrm{Hom}\Big(\Sigma\widetilde{H}_*(M^+;\mathbb{Q}), [\widetilde{H}_*(M^+;\mathbb{Q})^{\otimes 2}]_{\mathfrak{S}_2}\Big),$$

and so, describe $\Gamma$ by an extension

$$1 \to \mathrm{Hom}\big(\Sigma\widetilde{H}_*(M^+;\mathbb{Q}), S^2\big(\widetilde{H}_*(M^+;\mathbb{Q})\big)/\mathrm{Im}(\delta_*)\big) \to \Gamma \to \mathrm{Aut}\big(\widetilde{H}_*(M^+;\mathbb{Q}), \delta_*\big) \to 1.$$

This implies the following. We continue to assume that $M$ is even-dimensional and orientable. Let $G$ denote the group of homotopy classes of pointed homotopy self-equivalences of $M^+$ which act as the identity on $\widetilde{H}_*(M^+;\mathbb{Q})$.

**Theorem 2.2.** *If $M$ is orientable and even-dimensional, then $G$ acts on $H^*(C_n(M);\mathbb{Q})$ via* $\mathrm{Hom}(\Sigma\widetilde{H}_*(M^+;\mathbb{Q}), S^2(\widetilde{H}_*(M^+;\mathbb{Q}))/\mathrm{Im}(\delta_*))$.

**Example 2.3.** When $M$ is a punctured surface, one has $\widetilde{H}_*(M^+;\mathbb{Q}) = \Sigma H_1(M;\mathbb{Q}) \oplus \Sigma^2\mathbb{Q}$ so the map $\delta_* : \widetilde{H}_*(M^+;\mathbb{Q}) \to S^2(\widetilde{H}_*(M^+;\mathbb{Q}))$ has the form

$$\Sigma H_1(M;\mathbb{Q}) \oplus \Sigma^2\mathbb{Q} \longrightarrow \Sigma^2\Lambda^2(H_1(M;\mathbb{Q})) \oplus \Sigma^3 H_1(M;\mathbb{Q}) \oplus \Sigma^4\mathbb{Q},$$

which in grading 2 is the inclusion of the symplectic form $\omega \in \Lambda^2(H_1(M;\mathbb{Q}))$ and is zero otherwise. Thus, the above is $\mathrm{Hom}(H_1(M;\mathbb{Q}), \Lambda^2(H_1(M;\mathbb{Q}))/\langle\omega\rangle) \oplus H_1(M;\mathbb{Q})$. Using Poincaré duality and $\Lambda^2(H_1(M;\mathbb{Q})) \cong \mathbb{Q}\{\omega\} \oplus \Lambda^2(H_1(M;\mathbb{Q}))/\langle\omega\rangle$, this can be identified with $\mathrm{Hom}(H_1(M;\mathbb{Q}), \Lambda^2(H_1(M;\mathbb{Q})))$. This is the target of the Johnson homomorphism, cf. [24].

*Remark* 2.4. The results of this section should also follow from [24, Theorem 1.2] and some rational homotopy theory.

## 3 | PROOF OF THEOREM 1.2

Recall that $X \in \mathsf{Top}_*$ is *well based* if the basepoint map $i : * \to X$ is a closed cofibration: under this condition, $X \wedge -$ preserves weak equivalences between well-based spaces, and preserves closed cofibrations. Let us say that an $\mathbb{N}$-graded based space $Y$ is well based if $Y(n)$ is well based for each $n \in \mathbb{N}$.

Let us write $\mathbf{R} := \mathbf{Com}(L^+[1])$ and $\mathbf{S} := \mathbf{Com}([(L \oplus L)^+]_{\mathfrak{S}_2}[2])$ to ease notation, so $\Delta : \mathbf{S} \to \mathbf{R}$ makes $\mathbf{R}$ into a $\mathbf{S}$-module.

**Lemma 3.1.** *$\mathbf{S}$ and $\mathbf{R}$ are well based. The subspace of $[(L^+)^{\wedge p}]_{\mathfrak{S}_p}$ of those tuples which do not have distinct $M$ coordinates is well based, and this inclusion is a closed cofibration.*

*Proof.* Recall that $M$ is the interior of a compact manifold with boundary $\overline{M}$. This admits a collar, showing that $i : M \to \overline{M}$ admits a homotopy inverse, and so, the vector bundle $L \to M$ extends to a vector bundle over $\overline{M}$, which we also call $L$. Furthermore, choosing an inner product on this bundle, we can form the closed disc bundle $D(L) \to \overline{M}$, and consider $L$ as lying inside it as the open disc bundle. Now $D(L)$ is a manifold with boundary $\partial D(L) = S(L) \cup D(L)|_{\partial\overline{M}}$, and $L^+ = D(L)/\partial D(L)$.

Observe that $(\overline{M}, \partial\overline{M})$ is an compact manifold pair so (is a Euclidean neighbourhood retract pair and hence) can be expressed as a retract of a pair $(|X_\bullet|, |\partial X_\bullet|)$ of the geometric realisations of a simplicial set and a subset. We may pull $L$ back to $|X_\bullet|$ using the retraction; let us call this $L_X$. Now $D(L_X)/S(L_X) \cup D(L_X)|_{|\partial X_\bullet|}$ can be given an evident cell structure (by induction over the relative cells of $|\partial X_\bullet| \to |X_\bullet|$), and $L^+ = D(L)/\partial D(L)$ is a retract of it, so is well based. More generally, for the exterior direct sum $L_X^{\boxplus p} \to |X_\bullet^p|$ and writing $\partial|X_\bullet^p|$ for the subcomplex where some factor lies in $\partial X_\bullet$, there is a cell structure on $D(L_X^{\boxplus p})/S(L_X^{\boxplus p}) \cup D(L_X^{\boxplus p})|_{\partial|X_\bullet^p|}$ for which the group $\mathfrak{S}_p$ acts cellularly, and so, $[D(L_X^{\boxplus p})/S(L_X^{\boxplus p}) \cup D(L_X^{\boxplus p})|_{\partial|X_\bullet^p|}]_{\mathfrak{S}_p}$ is a cell complex of which $[(L^+)^{\wedge p}]_{\mathfrak{S}_p}$ is a retract, and so is well based. This shows that $\mathbf{R}$ is well based, and similar reasoning shows $\mathbf{S}$ is.

For the second statement,

$$inc : F := \text{fat diagonal of } |X_\bullet|^p = |\text{fat diagonal of } X_\bullet^p| \longrightarrow |X_\bullet^p| = |X_\bullet|^p$$

is the inclusion of a $\mathfrak{S}_p$-CW-subcomplex, and so has a $\mathfrak{S}_p$-equivariant open neighbourhood $U$ which equivariantly deformation retracts to it. This may be chosen to preserve the subcomplexes where some factor lies in $|\partial X_\bullet|$. Thus, it lifts to a $\mathfrak{S}_p$-equivariant deformation retraction of an open neighbourhood of $L_X^{\boxplus p}|_F \to L_X^{\boxplus p}$, and descends to the quotient by the subcomplexes where some factor lies in $|\partial X_\bullet|$. As it is equivariant, it descends further to the $\mathfrak{S}_p$-quotient. That is, it proves the claim for $(\overline{M}, \partial\overline{M}, L)$ replaced by $(|X_\bullet|, |\partial X_\bullet|, L_X)$; as the former data are a retract of the latter, the claim follows. □

**Lemma 3.2.** $\mathbf{R}$ *is a flat* $\mathbf{S}$*-module, in the sense that* $\mathbf{R} \otimes_{\mathbf{S}} -$ *preserves weak equivalences between left* $\mathbf{S}$*-modules whose underlying objects are well based.*

*Proof.* Recall that $\mathbf{R}(n) = [(L^+)^{\wedge n}]_{\mathfrak{S}_n}$. Define a filtration of $\mathbf{R}$ by $F_0\mathbf{R} = \mathbf{S}$ and

$$F_p\mathbf{R}(n) := F_{p-1}\mathbf{R}(n) \cup \mathrm{Im}\Big((L^+)^{\wedge p} \wedge ((L \oplus L)^+)^{\wedge(n-p)/2} \to \mathbf{R}(n)\Big),$$

where the latter term is only taken when it makes sense: for $n - p$ even. This is a filtration by right $\mathbf{S}$-modules. One checks that the diagram

$$\begin{array}{ccc} F_{p-2}\mathbf{R}(p)[p] \otimes \mathbf{S} & \longrightarrow & F_{p-1}\mathbf{R} \\ \downarrow & & \downarrow \\ \mathbf{R}(p)[p] \otimes \mathbf{S} & \longrightarrow & F_p\mathbf{R} \end{array}$$

is a pushout (in $\mathsf{Top}_*^{\mathbb{N}}$ and so in right $\mathbf{S}$-modules), where the horizontal maps are induced by the $\mathbf{S}$-module structure and the adjoints of the map $inc : F_{p-2}\mathbf{R}(p) \to F_{p-1}\mathbf{R}(p)$, and the map $id : \mathbf{R}(p) \to F_p\mathbf{R}(p)$.

We prove by induction on $p$ that $F_p\mathbf{R}$ is a flat $\mathbf{S}$-module in the indicated sense. As $F_0\mathbf{R} = \mathbf{S}$, these properties hold for $p = 0$. For $\mathbf{M}$, a left $\mathbf{S}$-module whose underlying object is well based, applying $- \otimes_{\mathbf{S}} \mathbf{M}$ to the square above gives a pushout square

$$\begin{array}{ccc} F_{p-2}\mathbf{R}(p)[p] \otimes \mathbf{M} & \longrightarrow & F_{p-1}\mathbf{R} \otimes_{\mathbf{S}} \mathbf{M} \\ \downarrow & & \downarrow \\ \mathbf{R}(p)[p] \otimes \mathbf{M} & \longrightarrow & F_p\mathbf{R} \otimes_{\mathbf{S}} \mathbf{M}. \end{array} \tag{3.1}$$

The map $F_{p-2}\mathbf{R}(p) \to \mathbf{R}(p)$ is the inclusion of the subspace of those $p$-tuples of points in $M$ labelled by $L$ which do not have distinct $M$ coordinates, so is a closed cofibration from a well-based space by the second part of Lemma 3.1. As $\mathbf{M}$ is assumed well based, the left-hand vertical map in (3.1) is a closed cofibration in each grading, and so, this square is also a homotopy pushout. A weak equivalence $f : \mathbf{M} \xrightarrow{\sim} \mathbf{M}'$ then induces a map of homotopy pushout squares which is a weak equivalence on all but the bottom right corner, by inductive assumption, so also induces a weak equivalence on this corner.

Thus, each $F_p\mathbf{R}$ is flat in the indicated sense, so $\mathbf{R}$ is too because $F_p\mathbf{R} \to \mathbf{R}$ is an isomorphism when evaluated on $n < p$, so $F_p\mathbf{R} \otimes_{\mathbf{S}} \mathbf{M} \to \mathbf{R} \otimes_{\mathbf{S}} \mathbf{M}$ is too. □

*Remark* 3.3. In the case that $L$ is the 0-dimensional vector bundle, the filtration stage $F_p\mathbf{R}(n)$ consists of those elements in the $n$th-based symmetric power $[(M^+)^{\wedge n}]_{\mathfrak{S}_n}$ containing at most $p$ unrepeated elements. Up to reindexing, this is the same as the filtration used by Arnol'd [1] and by Segal [23].

**Lemma 3.4.** *The induced map* $\mathbf{R} \otimes_{\mathbf{S}} S^0[0] \to \mathbf{C}(M; L)$ *is an isomorphism.*

*Proof.* By definition of the relative tensor product, there is a coequaliser diagram

$$\mathbf{R} \otimes \mathbf{S} \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \mathbf{R} \longrightarrow \mathbf{R} \otimes_{\mathbf{S}} S^0[0]$$

in $\mathsf{Top}_*^{\mathbb{N}}$, where $\alpha$ is given by the $\mathbf{S}$-module structure on $\mathbf{R}$, and $\beta$ is induced by the augmentation $\epsilon : \mathbf{S} \to S^0[0]$. The image of $\mathbf{R} \otimes \ker(\epsilon)(n) \to \mathbf{R}(n) = [(L^+)^{\wedge n}]_{\mathfrak{S}_n}$ is precisely the image of $(L^+)^{\wedge n-2} \wedge (L \oplus L)^+ \to [(L^+)^{\wedge n}]_{\mathfrak{S}_n}$, whose cofibre is by definition $\mathbf{C}(M; L)$. □

*Proof of Theorem* 1.2. Apply Lemma 3.2 to the weak equivalence $B(\mathbf{S}, \mathbf{S}, S^0[0]) \xrightarrow{\sim} S^0[0]$, giving an equivalence $B(\mathbf{R}, \mathbf{S}, S^0[0]) \xrightarrow{\sim} \mathbf{R} \otimes_{\mathbf{S}} S^0[0]$, and the latter is isomorphic to $\mathbf{C}(M)$ by Lemma 3.4. □

*Remark* 3.5. It is possible to fool oneself into thinking that the above argument can be adapted to the case of ordered configuration spaces, considered in the category of symmetric sequences of pointed spaces, in order to prove a statement analogous to the equivalence (1.2) in this category. Unfortunately, that statement is false. One can verify this directly in the case $M =*$ with trivial 0-dimensional Euclidean bundle, in grading 3. If there is an analogue for ordered configuration spaces, its statement must be more complicated.

## APPENDIX A: HOMOLOGICAL DENSITIES BY QUOC P. HO AND OSCAR RANDAL-WILLIAMS

### A.1 | Spaces of 0-cycles

It is easy to generalise Theorem 1.2 to the following variant of configuration spaces, called 'spaces of 0-cycles' by Farb–Wolfson–Wood [10]. Let $m, k \geqslant 1$, and for $n_1, n_2, \ldots, n_m \in \mathbb{N}$, let

$$Z^k_{n_1,\ldots,n_m}(M) \subset \mathrm{Sym}_{n_1,\ldots,n_m}(M) := [M^{n_1}]_{\mathfrak{S}_{n_1}} \times [M^{n_2}]_{\mathfrak{S}_{n_2}} \times \cdots \times [M^{n_m}]_{\mathfrak{S}_{n_m}}$$

be the open subspace of those $(\{x^1_1, \ldots, x^1_{n_1}\}, \{x^2_1, \ldots, x^2_{n_2}\}, \ldots, \{x^m_1, \ldots, x^m_{n_m}\})$ such that no $x^i_j$ has multiplicity $\geqslant k$ in all of these $m$ multisets. That is, $Z^k_{n_1,\ldots,n_m}(M)$ is the configuration space of particles of $m$ different colours, $n_i$ having colour $i$, which may collide except that no point of $M$ may carry $\geqslant k$ points of every colour. The one-point compactifications $Z_{n_1,\ldots,n_m}(M)^+$ again have a superposition product

$$Z^k_{n_1,\ldots,n_m}(M)^+ \wedge Z^k_{n'_1,\ldots,n'_m}(M)^+ \longrightarrow Z^k_{n_1+n'_1,\ldots,n_m+n'_m}(M)^+,$$

giving a commutative monoid $\mathbf{Z}^{m,k}(M)$ in $\mathbb{N}^m$-graded pointed spaces. Just as before, we can introduce labels in a vector bundle $L \to M$, giving $Z^k_{n_1,\dots,n_m}(M;L)$ and $\mathbf{Z}^{m,k}(M;L)$. Writing $1_i = (0,\dots,0,1,0\dots,0) \in \mathbb{N}^m$ with the 1 in the $i$th position, there is a pushout square

$$\begin{array}{ccc} \mathbf{Com}([(L^{\oplus mk})^+]_{\mathfrak{S}^m_k}[k,k,\dots,k]) & \xrightarrow{\epsilon} & S^0[0,\dots,0] \\ \downarrow{\scriptstyle\Delta} & & \downarrow \\ \mathbf{Com}\left(\bigvee_{i=1}^m L^+[1_i]\right) & \longrightarrow & \mathbf{Z}^{m,k}(M;L) \end{array} \tag{A.1}$$

of unital commutative monoids in $\mathsf{Top}_*^{\mathbb{N}^m}$, where $\Delta$ is now induced by the inclusion $[(L^{\oplus mk})^+]_{\mathfrak{S}^m_k} \to [(L^+)^{\wedge k}]_{\mathfrak{S}_k} \wedge \cdots \wedge [(L^+)^{\wedge k}]_{\mathfrak{S}_k} = \mathbf{Com}(\bigvee_{i=1}^m L^+[1_i])(k,\dots,k)$. The same argument as Theorem 1.2 shows that there is an equivalence

$$\mathbf{Com}(\vee_{i=1}^m L^+[1_i]) \otimes^{\mathbb{L}}_{\mathbf{Com}([(L^{\oplus mk})^+]_{\mathfrak{S}^m_k}[k,\dots,k])} S^0[0,\dots,0] \xrightarrow{\sim} \mathbf{Z}^{m,k}(M;L). \tag{A.2}$$

### A.2 | Revisiting homological densities

This can be used to revisit the work of Farb–Wolfson–Wood [10] and Ho [14] on homological densities, and, in particular, to explain coincidences of homological densities at the level of topology rather than algebra, as proposed in [14, 1.5.1].

The spaces $Z^k_{n_1,\dots,n_m}(M;L)$ are $\mathbb{Q}$-homology manifolds, being open subspaces of a product of coarse moduli spaces $[L^n]_{\mathfrak{S}_n}$ of the orbifolds $L^n /\!/ \mathfrak{S}_n$. As before, we suppose that $M$ is $d$-dimensional and take $L = W$ to be given by the sum of the orientation line of $M$ plus $(d-1)$ trivial lines: then the $Z^k_{n_1,\dots,n_m}(M;W)$ are orientable $\mathbb{Q}$-homology manifolds, of dimension $2d \cdot \sum n_i$. Again, they are vector bundles over $Z^k_{n_1,\dots,n_m}(M)$, so Poincaré duality gives

$$H^*\left(Z^k_{n_1,\dots,n_m}(M)\right) \cong H^*\left(Z^k_{n_1,\dots,n_m}(M;W)\right) \cong \widetilde{H}_{2d\sum n_i - *}\left(Z^k_{n_1,\dots,n_m}(M;W)^+\right).$$

On the other hand, the bar construction formula above together with the argument of Section 2.2 identifies the multigraded vector space $H_{*,*}(\mathbf{Z}^{m,k}(M;W))$ with

$$\mathrm{Tor}_*^{S^*(\widetilde{H}_*([(W^{\oplus mk})^+]_{\mathfrak{S}^m_k})[k,\dots,k])}\left(S^*\left(\bigoplus_{i=1}^m \widetilde{H}_*(W^+)[1_i]\right), \mathbb{Q}[0,\dots,0]\right).$$

#### A.2.1 | Odd-dimensional manifolds

As in Section 2.2, we have $\widetilde{H}_*([(W^{\oplus mk})^+]_{\mathfrak{S}^m_k}) \cong [\Sigma^{dmk}\widetilde{H}_*(M^+)]_{\mathfrak{S}^m_k}$ by the Thom isomorphism. If $d$ is odd, then the permutation group $\mathfrak{S}^m_k$ acts on the Thom class via the sign homomorphism $\mathfrak{S}^m_k \leqslant \mathfrak{S}_{mk} \to \mathbb{Z}^\times$, so acts nontrivially if $k \geqslant 2$ and trivially if $k = 1$. If $k \geqslant 2$, this means that $\widetilde{H}_*([(W^{\oplus mk})^+]_{\mathfrak{S}^m_k}) = 0$, showing that

$$H_{*,*}(\mathbf{Com}(\vee_{i=1}^m W^+[1_i])) \xrightarrow{\sim} H_{*,*}(\mathbf{Z}^{m,k}(M;W))$$

in this case. Using Poincaré duality on both sides gives [10, Theorem 1.4], except that that theorem is erroneously claimed for all $k \geqslant 1$. We will return to the case $k = 1$ below.

### A.2.2 | Even-dimensional manifolds

If $d$ is even, then $\mathfrak{S}_k^m$ acts trivially on $\Sigma^{dmk}\widetilde{H}_*(M^+)$, and using the Thom isomorphism to identify $\widetilde{H}_*(W^+) \cong \Sigma^d\widetilde{H}_*(M^+)$ too, the Koszul complex for computing the Tor-groups above is

$$\left(S^*\left(\bigoplus_{i=1}^m \Sigma^d\widetilde{H}_*(M^+;\mathbb{Q}^{w_1})[1_i] \oplus \Sigma^{dmk+1}\widetilde{H}_*(M^+;(\mathbb{Q}^{w_1})^{\otimes mk})[k,\dots,k]\right),\partial\right).$$

The differential $\partial$ is induced by the map

$$\Sigma^{dmk}\widetilde{H}_*(M^+;(\mathbb{Q}^{w_1})^{\otimes mk}) \to S^k(\Sigma^d\widetilde{H}_*(M^+;\mathbb{Q}^{w_1})) \otimes \cdots \otimes S^k(\Sigma^d\widetilde{H}_*(M^+;\mathbb{Q}^{w_1}))$$

obtained by linearly dualising the cup product map

$$H_c^*(M;\mathbb{Q}^{w_1})^{\otimes mk} \longrightarrow H_c^*(M;(\mathbb{Q}^{w_1})^{\otimes mk}), \tag{A.3}$$

and so is trivial if (and only if) all $mk$-fold cup products of ($w_1$-twisted) compactly supported cohomology classes on $M$ vanish.

When this cup product map is trivial, so $\partial$ is trivial, the above just gives a formula for $H_{*,*}(\mathbf{Z}^{m,k}(M;W))$. Using Poincaré duality, and reindexing, to express this in terms of $H^*(Z^k_{n_1,\dots,n_m}(M))$ and $H^*(\mathrm{Sym}_{n_1,\dots,n_m}(M))$, we obtain an identity of multigraded vector spaces

$$H^*(Z^k_\bullet(M)) \cong H^*(\mathrm{Sym}_\bullet(M)) \otimes S^*(\Sigma^{d(mk-1)-1}H^*(M;(\mathbb{Q}^{w_1})^{\otimes mk-1})[k,\dots,k]).$$

There are stabilisation maps $\sigma_i : H^*(Z^k_{n_1,\dots,n_m}(M)) \to H^*(Z^k_{n_1,\dots,n_i+1,\dots,n_m}(M))$ analogous to those constructed in Section 2.3, similarly for $H^*(\mathrm{Sym}_{n_1,\dots,n_m}(M))$, and both stabilise as $n_j \to \infty$, just as in Section 2.3: this recovers [10, Theorem 1.7]. We may take the colimit of all these stabilisations to obtain

$$H^*(Z^k_{\infty,\dots,\infty}(M)) \cong H^*(\mathrm{Sym}_{\infty,\dots,\infty}(M)) \otimes S^*(\Sigma^{d(mk-1)-1}H^*(M;(\mathbb{Q}^{w_1})^{\otimes mk-1})).$$

Writing $P_{Z^{m,k}}(t)$ and $P_{Sym^m}(t)$ for the Poincaré series of $H^*(Z^k_{\infty,\dots,\infty}(M))$ and $H^*(\mathrm{Sym}_{\infty,\dots,\infty}(M))$, respectively, this discussion identifies the *homological density* $P_{Z^{m,k}}(t)/P_{Sym^m}(t)$ with the Poincaré series of $S^*(\Sigma^{d(mk-1)-1}H^*(M;(\mathbb{Q}^{w_1})^{\otimes mk-1}))$. This visibly only depends on the product $mk$, giving 'coincidences between homological densities': this recovers [10, Theorem 1.2]; in fact, it also recovers the stronger Theorem 3.6 of that paper.

### A.2.3 | Odd-dimensional manifolds, $k = 1$

Just as in the even-dimensional case, if the cup product map (A.3) is zero, then one gets an explicit description of $H^*(Z_\bullet(M))$, and the homological density is given by the Poincaré series of the graded vector space $S^*(\Sigma^{d(m-1)-1}H^*(M;(\mathbb{Q}^{w_1})^{\otimes m-1}))$. It follows from Section A.2.1 that the homological density is 1 for $k > 1$, so for odd-dimensional manifolds, it is *not true* that the homological density depends only on $mk$.

### A.2.4 | Euler characteristic

If the cup product map (A.3) is not zero, and either $d$ is even or $d$ is odd and $k = 1$, then there is instead a nontrivial differential on the multigraded vector space

$$H^*(\mathrm{Sym}_\bullet(M)) \otimes S^*(\Sigma^{d(mk-1)-1}H^*(M;(\mathbb{Q}^{w_1})^{\otimes mk-1})[k,\dots,k]),$$

of degree $(+1,0)$, whose homology is $H^*(Z^k_{*,\dots,*}(M))$. Then one would not expect $\frac{P_Z(t)}{P_{Sym}(t)}$ to agree with the Poincaré series of $S^*(\Sigma^{d(mk-1)-1}H^*(M;(\mathbb{Q}^{w_1})^{\otimes mk-1}))$, and indeed, it does not [10, Remark 1.6]. However, as Euler characteristic commutes with taking homology, we have the identity

$$\sum_{n_1,\dots,n_m\geqslant 0} \chi\left(Z^k_{n_1,\dots,n_m}(M)\right) s_1^{n_1}\cdots s_m^{n_m} = \left(\prod_{i=1}^m (1-s_i)\right)^{-\chi(M)} \cdot (1-(s_1\cdots s_m)^k)^{\chi(M,(\mathbb{Q}^{w_1})^{\otimes mk-1})}$$

in $\mathbb{Z}[[s_1,\dots,s_m]]$. The left-hand factor is $\sum_{n_1,\dots,n_m\geqslant 0}\chi(\mathrm{Sym}_{n_1,\dots,n_m}(M))s_1^{n_1}\cdots s_m^{n_m}$. This recovers [10, Theorem 1.9 1].

## A.3 | Spectral densities

The construction of homological densities can be promoted to the level of spectra, addressing [14, 1.5.1], as follows. Let us *assume that $M$ is even-dimensional and orientable*: then we can dispense with twisting by the vector bundle $W \to M$. We consider $\mathbf{Z}^{m,k}(M)$ with its $\mathbb{N}^m$-grading reduced to an $\mathbb{N}$-grading via sum : $\mathbb{N}^m \to \mathbb{N}$. Collapsing the complement of a small neighbourhood of a point in $M$ gives a map $M^+ \to S^d$, inducing a map of commutative monoids

$$\mathbf{Com}\left(\bigvee_{i=1}^m M^+[1]\right) \longrightarrow \mathbf{Com}(S^d[1]).$$

If $X$ is a left $\mathbf{Com}(S^d[1])$-module, it is equipped with maps $S^d \wedge X(n) \to X(n+1)$, and so, we can define the spectrum $\overline{X} := \mathrm{hocolim}_{n\to\infty} S^{-nd} \wedge \Sigma^\infty X(n)$. Using these two constructions, we may therefore form the spectrum

$$\Delta^{m,k} := \overline{\mathbf{Com}(S^d[1]) \otimes^{\mathbb{L}}_{\mathbf{Com}(\bigvee_{i=1}^m M^+[1])} \mathbf{Z}^{m,k}(M)}.$$

By analogy with [14, Section 7.5], we propose $\Delta^{m,k}$ as a spectral form of the stable density of $Z^k_{n_1,\dots,n_m}(M)$ in $\mathrm{Sym}_{n_1,\dots,n_m}(M)$. At the level of $\mathbb{Q}$-chains, it recovers the construction from the proof of Theorem 7.5.1 of [14]. We can prove the spectral form of that theorem analogously: as $\mathbb{N}$-graded objects, there is an evident map from (A.1) to the analogous square for $\mathbf{Z}^{1,mk}(M)$ which induces a map of spectra $\Delta^{m,k} \to \Delta^{1,mk}$, and this is an equivalence by (A.2) as both are identified with $\overline{\mathbf{Com}(S^d[1]) \otimes^{\mathbb{L}}_{\mathbf{Com}(M^+[mk])} S^0[0]}$.

This may be simplified for $mk \geqslant 2$ as follows. The map $M^+ \to [(S^d)^{\wedge mk}]_{\mathfrak{S}_{mk}}$ with which the derived tensor product is formed factors over $(S^d)^{\wedge mk}$ so is nullhomotopic when $mk \geqslant 2$, and so, $\mathbf{Com}(S^d[1]) \otimes^{\mathbb{L}}_{\mathbf{Com}(M^+[mk])} S^0[0]$ is equivalent to $\mathbf{Com}(S^d[1]) \otimes (S^0[0] \otimes^{\mathbb{L}}_{\mathbf{Com}(M^+[mk])} S^0[0])$ as a left $\mathbf{Com}(S^d[1])$-module. In this situation, the $\overline{(-)}$ construction gives

$$\Delta^{m,k} \simeq \bigvee_{n\geqslant 0} S^{-nd} \wedge \Sigma^\infty(S^0[0] \otimes^{\mathbb{L}}_{\mathbf{Com}(M^+[mk])} S^0[0])(n)$$

$$\simeq \bigvee_{n\geqslant 0} S^{-nd} \wedge \Sigma^\infty \mathbf{Com}(S^1 \wedge M^+[mk])(n).$$

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