RELATIVE SERRE DUALITY FOR HECKE CATEGORIES

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ABSTRACT. We prove a conjecture of Gorsky, Hogancamp, Mellit, and Nakagane in the Weyl group case. Namely, we show that the left and right adjoints of the parabolic induction functor between the associated Hecke categories of Soergel bimodules differ by the relative full twist.

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1. INTRODUCTION

1.1. **Soergel bimodules.** Let G be a connected reductive group over $\overline{\mathbb{F}}_q$, equipped with a fixed Borel subgroup B and a fixed maximal torus $T \subseteq B$. Let $R := \mathsf{Sym}(X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}\langle -2 \rangle)$ be the graded polynomial algebra generated by $\operatorname{rank} T$ elements where the generators live in graded degree 2. Here, the angular bracket $\langle - \rangle$ denotes a formal grading shift, which is distinct from the square bracket $[-]$ (which will appear later on in the paper) used to denote a cohomological shift.

Consider $\mathsf{BiMod}_R(\mathsf{Vect}^{\mathrm{gr},\heartsuit})$, the monoidal abelian category of graded bimodules over R, where the monoidal product $\otimes_R$ is denoted by $\star$. By construction, R is equipped with an action of the Weyl group W of G. The category of Soergel bimodules[1] SBim_W is the full idempotent complete monoidal additive subcategory of $\mathsf{BiMod}_R(\mathsf{Vect}^{\mathrm{gr},\heartsuit})$ stable under grading shifts and generated by objects of the form $R \otimes_{R^s} R$ where $s \in W$ is a simple reflection. In other words, SBim_W is generated, under taking finite direct sums, summands, and grading shifts, by $R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \cdots \otimes_{R^{s_k}} R$ for any sequence of simple reflections $s_i \in W$.

Let $\mathsf{Ch}^b(\mathsf{SBim}_W)$ denote the monoidal DG-category of bounded chain complexes of Soergel bimodules, whose monoidal product is also denoted by $\star$. In [R], Rouquier constructs an object $R_\beta \in \mathsf{Ch}^b(\mathsf{SBim}_W)$,

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[1] The notation SBim_W is slightly abusive as SBim_W depends on the Coxeter system and not just the Weyl group W. Note also that, as the notation might suggest, the category SBim_W can be more generally be defined for any Coxeter system. In this paper, we will only consider the (finite) Weyl group case.

known as the Rouquier complex, associated to each $\beta \in \mathrm{Br}_W$, the corresponding braid group, that is compatible with the braid relations in the sense that we have an equivalence of objects in $\mathsf{Ch}^b(\mathsf{SBim}_W)$

$$R_{\beta_1} \star R_{\beta_2} \simeq R_{\beta_1\beta_2}, \qquad \text{for } \beta_1, \beta_2 \in \mathrm{Br}_W.$$

In particular, we have a complex $\mathsf{FT}_G \in \mathsf{Ch}^b(\mathsf{SBim}_W)$ associated to the full twist braid, i.e., the square of the longest element.

1.2. **Parabolic induction and restriction functors.** Let P be a proper standard parabolic subgroup of G with Levi factor L. The box tensor product induces a monoidal fully faithful embedding

$$\iota : \mathsf{Ch}^b(\mathsf{SBim}_{W_L}) \hookrightarrow \mathsf{Ch}^b(\mathsf{SBim}_W).$$

One can show that ι admits both a left and a right adjoint, denoted by ι^L and ι^R, respectively.[2] Since ι is a monoidal fully faithful embedding, unless confusion is likely to occur, we will identify objects in $\mathsf{Ch}^b(\mathsf{SBim}_{W_L})$ with their images in $\mathsf{Ch}^b(\mathsf{SBim}_W)$ via ι without explicitly invoking ι.

1.3. **Serre duality for Hecke categories.** Despite their representation theoretic origin, the categories of Soergel bimodules $\mathsf{Ch}^b(\mathsf{SBim}_W)$ in type A play an important role in low dimensional topology. For example, they are originally used in [K] to define the HOMFLY-PT homology of links and have since attracted a lot of attention in the study of link invariants.

Also in type A, and $P = B$, motivated by a certain symmetry in the HOMFLY-PT homology theory of links, Gorsky, Hogancamp, Mellit, and Nakagane showed that ι^L and ι^R are related to each other by the full twist FT_G. More precisely, they proved the following theorem, which refines some results of [BBM; MS] in these cases.

Theorem 1.3.1 ([G+]). *For $G = \mathrm{GL}_n$, P the parabolic subgroup given by the partition $(r, 1, 1, \dots, 1)$ (and hence, $L \simeq \mathrm{GL}_r \times \mathbb{G}_m^{n-r}$), we have a natural equivalence of functors $\iota^R \simeq \iota^L(\mathsf{FT}_{G,L} \star -)$, where $\mathsf{FT}_{G,L} := \mathsf{FT}_L^{-1} \star \mathsf{FT}_G$.*

This result is quite similar to the classical Verdier/Serre duality in algebraic geometry where the two types of pullbacks differ by the dualizing sheaf for a smooth morphism. The authors of [G+] thus refer to this result as a Serre duality for Hecke categories, where $\mathsf{FT}_{G,L}$ plays the role of the dualizing sheaf.

1.4. **Main result.** The main result of this paper is the following theorem, which generalizes Theorem 1.3.1 above to arbitrary connected reductive groups G and arbitrary parabolic subgroups. This was given as [G+, Conjecture 1.8] in their original paper.

Theorem 1.4.1. *We have an equivalence of functors $\iota^R \simeq \iota^L(\mathsf{FT}_{G,L} \star -)$.*

We will prove Theorem 1.4.1 by geometric means using a geometric avatar of $\mathsf{Ch}^b(\mathsf{SBim}_W)$. We expect that the general strategy of the proof can be adapted directly to the more combinatorial setup of Soergel bimodules. However, the geometric setup allows for a very efficient and transparent proof.

2. Geometric Hecke categories

In this section, we will describe the geometric setup and explain how it is related to the algebraic setup involving Soergel bimodules described in the introduction.

[2] See [G+] for more details or §2.2 below for a geometric perspective.

2.1. **Geometric setup.** We define the finite Hecke category associated to G to be

$$\mathsf{H}_G := \mathsf{Shv}_{\mathrm{gr},c}(B\backslash G/B),$$

where $\mathsf{Shv}_{\mathrm{gr},c}(B\backslash G/B)$ is the category of graded sheaves on $B\backslash G/B$ developed [HLa]. We recall that the theory of graded sheaves is defined for any Artin stack of finite type and affine stabilizers and is equipped with a six-functor formalism that formally behaves like the classical six-functor formalism for constructible ℓ-adic sheaves. In fact, for any such stack $\mathcal{Y}$, we have a functor of forgetting the grading

$$\mathrm{oblv}_{\mathrm{gr}} : \mathsf{Shv}_{\mathrm{gr},c}(\mathcal{Y}) \to \mathsf{Shv}_c(\mathcal{Y})$$

that realizes $\mathsf{Shv}_{\mathrm{gr},c}(\mathcal{Y})$ as a graded lift of $\mathsf{Shv}_c(\mathcal{Y})$. Moreover, $\mathrm{oblv}_{\mathrm{gr}}$ is compatible with the six-functor formalism on both sides.

In this paper, we only use the formal aspects of the theory of graded sheaves.

2.1.1. H_G is a monoidal category with respect to the convolution product $\star$. More precisely, if we let $\mathsf{Vect}^{\mathrm{gr}}$ be the symmetric monoidal stable ∞-category of graded chain complexes of $\overline{\mathbb{Q}}_\ell$-vector spaces and $\mathsf{Vect}^{\mathrm{gr},c}$ the full symmetric monoidal subcategory spanned by compact objects, i.e., those with finite-dimensional cohomology, supported in finitely many graded and cohomological degrees, then, H_G is an algebra object in $\mathsf{Vect}^{\mathrm{gr},c}$-Mod, the symmetric monoidal category of $\mathsf{Vect}^{\mathrm{gr},c}$-module categories. In particular, this means that the convolution product $\star$ is compatible with cohomological shifts $[-]$ and grading shifts $\langle - \rangle$.

When confusion is unlikely to arise, we will drop the $\star$, and write, for example KL instead of $K \star L$ for $K, L \in \mathsf{H}_G$. We have the following result, which allows us to work purely geometrically when studying $\mathsf{Ch}^b(\mathsf{SBim}_W)$.

Theorem 2.1.2 ([HLa, Theorem 4.4.1])**.** *We have an equivalence of monoidal categories*[3]

$$\mathsf{Shv}_{\mathrm{gr},c}(B\backslash G/B) \simeq \mathsf{Ch}^b(\mathsf{SBim}_W).$$

In this paper, we will work exclusive in the geometric setting, viewing all objects in $\mathsf{Ch}^b(\mathsf{SBim}_W)$ as objects in H_G.

Remark 2.1.3. The geometric version H_G plays an important role in low dimensional topology. For example, they are originally used in [WW] to geometrically define the HOMFLY-PT homology of links and used in [HLb] to establish a relation between the HOMFLY-PT link homology and Hilbert schemes of points on $\mathbb{C}^2$, as conjectured by Gorsky, Neguț, and Rasmussen in [GNR]. Note also that H_G is denoted as $\mathsf{H}_G^{\mathrm{gr}}$ in [HLb].

2.1.4. *Standard and co-standard objects.* By the Bruhat decomposition, we have a stratification of $B\backslash G/B$ by $B\backslash BwB/B$ for $w \in W$. Let $\jmath_w : B\backslash BwB/B \to B\backslash G/B$ denote the embedding. The standard (resp. co-standard) object Δ_w (resp. ∇_w) is defined to be $\jmath_{w!}\overline{\mathbb{Q}}_\ell$ (resp. $\jmath_{w*}\overline{\mathbb{Q}}_\ell$). By construction, we always have a map

$$\Delta_w \simeq \jmath_{w!}\overline{\mathbb{Q}}_\ell \to \jmath_{w*}\overline{\mathbb{Q}}_\ell \simeq \nabla_w. \tag{2.1.5}$$

Under the equivalence stated in Theorem 2.1.2, Δ_w (resp. ∇_w) corresponds to the Rouquier complex R_β associated to the positive (resp. negative) braid β associated to w. In particular, the full twist element FT_G corresponds to $\Delta_{w_0}^2$, where w_0 is the longest element of W.

[3] Since the graded sheaf theory $\mathsf{Shv}_{\mathrm{gr},c}(-)$ developed in [HLa] is based on the theory of ℓ-adic sheaves, it has coefficients in $\overline{\mathbb{Q}}_\ell$. Since $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$ as fields, there is no difference between working over $\mathbb{C}$ and $\overline{\mathbb{Q}}_\ell$.

2.2. **Geometric parabolic induction and restriction functors.** We will now describe the geometric version of the parabolic induction and restriction functors. Let P be a proper standard parabolic subgroup of G with Levi factor L. Let B_L be the Borel subgroup of L defined as the image of B in L. Consider the following correspondence

$$\begin{array}{ccc} & B\backslash P/B & \\ \swarrow^{p} & & \searrow^{q} \\ B_L\backslash L/B_L & & B\backslash G/B. \end{array}$$

2.2.1. *Parabolic induction functor.* Let $\iota := q_! p^* : \mathsf{H}_L \to \mathsf{H}_G$ denote the functor of parabolic induction. It is easy to see that this is a monoidal functor.

Note that p^* and p_* are inverses of each other

$$\mathsf{Shv}_{\mathrm{gr},c}(B_L\backslash L/B_L) \underset{p_*}{\overset{p^*}{\underset{\simeq}{\rightleftarrows}}} \mathsf{Shv}_{\mathrm{gr},c}(B\backslash P/B)$$

since p is a bundle with fiber BU_P, the classifying space of U_P, which is the unipotent radical of P. Since q is a closed embedding, $q_! \simeq q_*$ is fully faithful. Thus, ι also fully faithful. We will therefore frequently view objects of H_L as objects of H_G without explicitly invoking the functor ι. Note that by fully faithfulness, $\iota^R\iota \simeq \iota^L\iota \simeq \mathrm{id}_{\mathsf{H}_L}$.

2.2.2. *Parabolic restriction functors.* The functor ι admits a right adjoint, given by $\iota^R := p_* q^!$. Moreover, since q is proper and p_* and p^* are mutually inverses, ι also admits a left adjoint, given by $\iota^L := p_* q^*$.

3. The proof

We are now ready to prove Theorem 1.4.1.

3.1. **The strategy.** Theorem 1.4.1 is a consequence of Propositions 3.1.1 and 3.1.2.

Proposition 3.1.1. *The functor* $\mathsf{FT}_{G,L} \star -$ *induces an equivalence of categories* $\ker \iota^R \xrightarrow{\simeq} \ker \iota^L$.

Proposition 3.1.2. *There exists a morphism* $\alpha : \mathsf{FT}_G \to \mathsf{FT}_L$ *such that* $\iota^L(\alpha) : \iota^L(\mathsf{FT}_G) \to \iota^L(\mathsf{FT}_L) \simeq \mathsf{FT}_L$ *is an equivalence.*

We will now prove Theorem 1.4.1 assuming Propositions 3.1.1 and 3.1.2.

Proof of Theorem 1.4.1. Let $j : B\backslash(G-P)/B \to B\backslash G/B$ denote the complement of the closed immersion q. For any $K \in \mathsf{H}_G$, we have the following exact triangle

$$q_! q^! K \to K \to j_* j^* K$$

which is equivalent to

$$q_! p^* p_* q^! K \to K \to j_* j^* K$$

and hence, to

$$\iota\iota^R K \to K \to j_* j^* K.$$

Applying $\mathsf{FT}_{G,L} \star -$ to the above triangle, we obtain

$$\mathsf{FT}_{G,L}\iota\iota^R K \to \mathsf{FT}_{G,L} K \to \mathsf{FT}_{G,L} j_* j^* K.$$

Since $j_* j^* K \in \ker \iota^R$, we get $\mathsf{FT}_{G,L} j_* j^* K \in \ker \iota^L$, by Proposition 3.1.1. Thus, applying ι^L to the above triangle, we obtain

$$\iota^L(\mathsf{FT}_{G,L} K) \simeq \iota^L(\mathsf{FT}_{G,L}\iota\iota^R K). \tag{3.1.3}$$

But now,

$$\iota^L(\mathsf{FT}_{G,L}\iota\iota^R K) \simeq \iota^L(\mathsf{FT}_{G,L})\iota^R(K) \simeq \mathsf{FT}_L^{-1}\iota^L(\mathsf{FT}_G)\iota^R(K) \xrightarrow[\simeq]{\mathsf{FT}_L^{-1}\iota^L(\alpha)\iota^R(K)} \mathsf{FT}_L^{-1}\mathsf{FT}_L\iota^R(K) \simeq \iota^R(K). \tag{3.1.4}$$

Here, the first and second equivalences follow from the fact that H_L is rigid (see [HLb, Proposition 2.10.2]), which implies that ι^L is H_L-linear with respect to the actions of H_L on itself and on H_G via ι. The third equivalence follows from Proposition 3.1.2.

Combining (3.1.3) and (3.1.4) and observing that all the morphisms involved are natural in K, we obtain the desired equivalence. □

We will prove Propositions 3.1.1 and 3.1.2 in the remainder of the paper.

3.2. **Proof of Proposition 3.1.1.** We start with the following lemma.

Lemma 3.2.1. *The following statements are equivalent:*

(i) $\mathsf{FT}_{G,L} \star -$ *induces an equivalence of categories* $\ker \iota^R \xrightarrow{\simeq} \ker \iota^L$.

(ii) $\mathsf{FT}_G \star -$ *induces an equivalence of categories* $\ker \iota^R \xrightarrow{\simeq} \ker \iota^L$.

Proof. We will show that (ii) implies (i). The converse is similar.

By rigidity of H_L, which implies that ι^L and ι^R are H_L-linear, we see that $\mathsf{FT}_L \star -$ induces an equivalence of categories between $\ker \iota^R$ (resp. $\ker \iota^L$) with itself. But now, this observation, combined with the current assumption that $\mathsf{FT}_G \star -$ induces an equivalence between $\ker \iota^R$ and $\ker \iota^L$, allows us to conclude since

$$\mathsf{FT}_{G,L} \star - \simeq \mathsf{FT}_L^{-1} \star \mathsf{FT}_G \star -.$$

□

It remains to prove Lemma 3.2.1.(ii).

Lemma 3.2.2. $\mathsf{FT}_G \star -$ *induces an equivalence of categories* $\ker \iota^R \xrightarrow{\simeq} \ker \iota^L$.

Proof. Observe that

$$\ker \iota^R = \langle \nabla_u \mid u \notin W_L \rangle,$$

where the RHS denotes the smallest $\mathrm{Vect}^{\mathrm{gr},c}$-linear full stable infinity subcategory of H_G containing all objects of the form ∇_u for $u \notin W_L$. In other words, it is the smallest full subcategory of H_G containing ∇_u for $u \notin W_L$ that is closed under finite direct sums, shifts, cones, and grading shifts.

Since for any $u \in W$,

$$\ell(w_0 u) + \ell(u^{-1}) = \ell(w_0) - \ell(u) + \ell(u^{-1}) = \ell(w_0),$$

$\Delta_{w_0} \simeq \Delta_{w_0 u}\Delta_{u^{-1}} \simeq \Delta_{w_0 u}(\nabla_u)^{-1}$, or equivalently, $\Delta_{w_0}\nabla_u \simeq \Delta_{w_0 u}$. Thus,

$$\Delta_{w_0} \ker \iota^R = \langle \Delta_{w_0 u} \mid u \notin W_L \rangle = \langle \Delta_t \mid t \in \tau \rangle$$

where

$$\tau = \{ w_0 u \mid u \notin W_L \} \subset W.$$

Observe that τ is closed with respect to the Bruhat order on W, or equivalently, the union of $B\backslash BvB/B$ for $v \in \tau$ is a closed substack of $B\backslash G/B$. Indeed, since the map $W \to W$ given by $u \mapsto w_0 u$ reverses the Bruhat order (see [H, Example 3, p. 119]), to show that τ is closed, it suffices to show that $W \setminus W_L$ is open. But this is equivalent to the fact that W_L is closed, which is true.

Thus,

$$\Delta_{w_0} \ker \iota^R = \langle \Delta_t \mid t \in \tau \rangle = \langle \nabla_t \mid t \in \tau \rangle = \langle \nabla_{w_0 u} \mid u \notin W_L \rangle.$$

But now, we have

$$\begin{aligned} \Delta_{w_0}^2 \ker \iota^R &= \langle \Delta_{w_0} \nabla_{w_0 u} \mid u \notin W_L \rangle \\ &= \langle \Delta_{w_0^2 u} \mid u \notin W_L \rangle \\ &= \langle \Delta_u \mid u \notin W_L \rangle \\ &= \ker \iota^L, \end{aligned}$$

and we are done. □

The proof of Proposition 3.1.1 is complete. □

3.3. Proof of Proposition 3.1.2.

3.3.1. *Constructing α.* Observe that we can write $w_0 = uw_{0,L}$ such that $\ell(w_0) = \ell(u) + \ell(w_{0,L})$. Indeed, take $u = w_0 w_{0,L}^{-1}$, then since w_0 is the longest element

$$\ell(u) = \ell(w_0) - \ell(w_{0,L}^{-1}) = \ell(w_0) - \ell(w_{0,L})$$

and hence,

$$\ell(w_0) = \ell(u) + \ell(w_{0,L}).$$

Thus, we have

$$\mathsf{FT}_G \simeq \Delta_{w_0}^2 \simeq \Delta_{w_0}\Delta_{w_0^{-1}} \simeq \Delta_u \Delta_{w_{0,L}} \Delta_{w_{0,L}^{-1}} \Delta_{u^{-1}} \simeq \Delta_u \mathsf{FT}_L \Delta_{u^{-1}}.$$

Thus,

$$\mathsf{FT}_L \simeq \Delta_u^{-1}\mathsf{FT}_G \Delta_{u^{-1}}^{-1} \simeq \mathsf{FT}_G \Delta_u^{-1}\Delta_{u^{-1}}^{-1} \simeq \mathsf{FT}_G \nabla_{u^{-1}} \nabla_u, \tag{3.3.2}$$

where the second to last equivalence is due to the fact that FT_G commutes with Δ_u^{-1} since this is an identity in the braid group, see [SB, Satz 7.2].[4]

Now, the map α can be constructed as the composition

$$\mathsf{FT}_G \simeq \mathsf{FT}_G \Delta_u^{-1}\Delta_u \simeq \mathsf{FT}_G \nabla_{u^{-1}}\Delta_u \xrightarrow{\beta} \mathsf{FT}_G \nabla_{u^{-1}}\nabla_u \simeq \mathsf{FT}_L. \tag{3.3.3}$$

Here, the non-trivial morphism β is given by (2.1.5) and the last equivalence is due to (3.3.2).

3.3.4. *$\iota^L(\alpha)$ is an equivalence.* To show that $\iota^L(\alpha)$ is an equivalence, by construction, it suffices to show that the map $\iota^L(\beta)$ is an equivalence, where β is defined in (3.3.3). Equivalently, we want to show that $\mathrm{Cone}(\beta) \in \ker \iota^L$, which is equivalent to showing that

$$\mathsf{FT}_G \nabla_{u^{-1}} \mathrm{Cone}(\Delta_u \to \nabla_u) \in \ker \iota^L.$$

By Lemma 3.2.2, this is equivalent to

$$\nabla_{u^{-1}} \mathrm{Cone}(\Delta_u \to \nabla_u) \in \ker \iota^R,$$

which is a direct consequence of the following two lemmas using the fact that $i^! j_* \simeq 0$ where i and j form a pair of closed embedding and open complement.

Lemma 3.3.5. $\nabla_{u^{-1}} \mathrm{Cone}(\Delta_u \to \nabla_u) \in \langle \nabla_{u^{-1}v} \mid u > v \rangle$.

Lemma 3.3.6. $\{u^{-1}v \mid u > v\} \cap W_L = \emptyset$.

We will prove these two lemmas in the remainder of this subsection. For the proof of Lemma 3.3.5, we need the following elementary lemma.

Lemma 3.3.7. *For $x, y \in W$ such that $y > x$, we have*

$$\nabla_{y^{-1}}\Delta_x = \nabla_{y^{-1}x}.$$

Proof. Since $y > x$, we can write $y = xz$ such that $\ell(y) = \ell(x) + \ell(z)$ and hence, $y^{-1} = z^{-1}x^{-1}$, where $\ell(y^{-1}) = \ell(z^{-1}) + \ell(x^{-1})$. Thus,

$$\nabla_{y^{-1}}\Delta_x \simeq \nabla_{z^{-1}}\nabla_{x^{-1}}\Delta_x \simeq \nabla_{z^{-1}} \simeq \nabla_{y^{-1}x}.$$

□

Proof of Lemma 3.3.5. By construction,

$$\mathrm{Cone}(\Delta_u \to \nabla_u) \in \langle \Delta_v \mid u > v \rangle.$$

Thus, by Lemma 3.3.7,

$$\nabla_{u^{-1}} \mathrm{Cone}(\Delta_u \to \nabla_u) \in \langle \nabla_{u^{-1}v} \mid u > v \rangle.$$

□

[4]Note that this is strictly weaker than the centrality of FT_G as proved in [BT].

Proof of Lemma 3.3.6. Recall that $u = w_0 w_{0,L}^{-1} = w_0 w_{0,L}$ and hence, $u^{-1}v = w_{0,L} w_0 v$. Thus, it suffices to show that $w_0 v \notin W_L$ for all $v < u$. To do that, we will show that for such v, $\ell(w_0 v) > \ell(w_{0,L})$, which implies the desired result since $w_{0,L}$ is the longest element in W_L. This is true since $\ell(u) > \ell(v)$, and hence $\ell(w_0 v) > \ell(w_0 u) = \ell(w_{0,L})$. □

The proof of Proposition 3.1.2 is now complete. □

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